# Local: Metric tensor

local spacetime line element

$$\mathrm{d}s^2 = \mathrm{d}(ct)^2 - [\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2]$$

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$= \mathrm{d}(ct)^2 - \mathrm{d}r^2 - r^2[\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2]$$

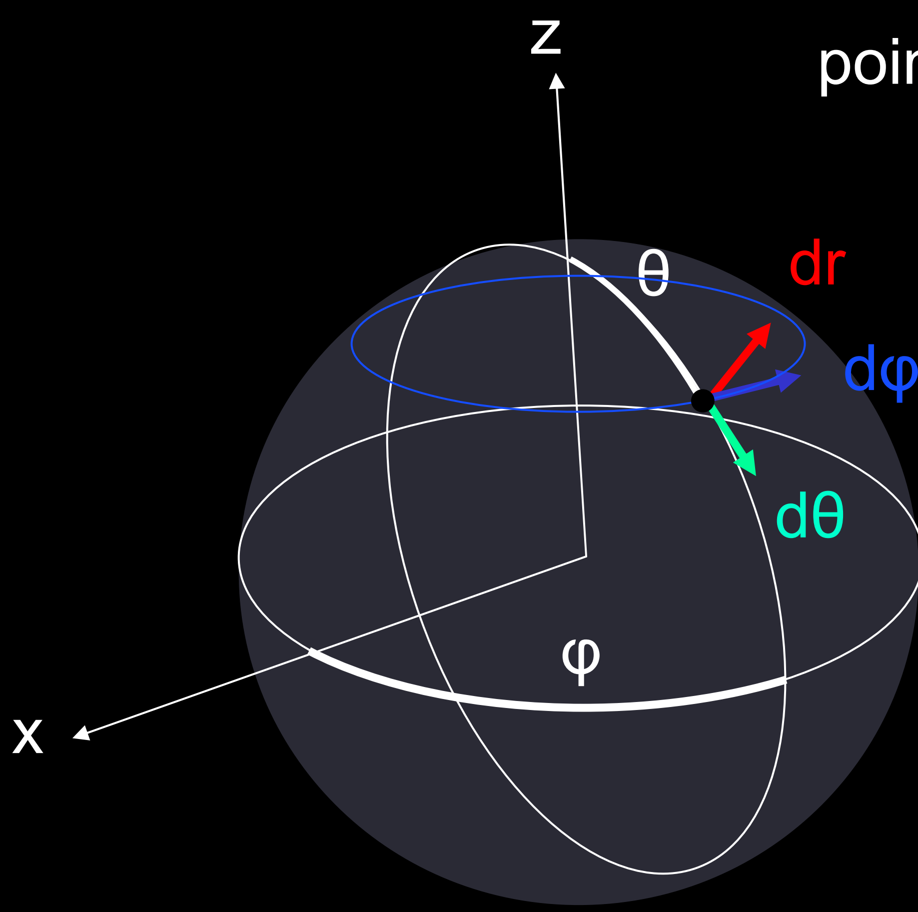
$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix}$$

$$= \mathrm{d}(ct)^2 - R^2\left[\mathrm{d}\chi^2 + \chi^2[\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2]\right]$$

$$r = R\chi \quad (R = \mathrm{const})$$

Minkowski spacetime, Euklidean (flat) space

# Polar coordinates




point $(r, \theta, \varphi)$

$d\varphi$ (length=$r \sin \theta \, d\varphi$)

$d\theta$ (length=$r \, d\theta$)

Minkowski spacetime, Euklidean (flat) space

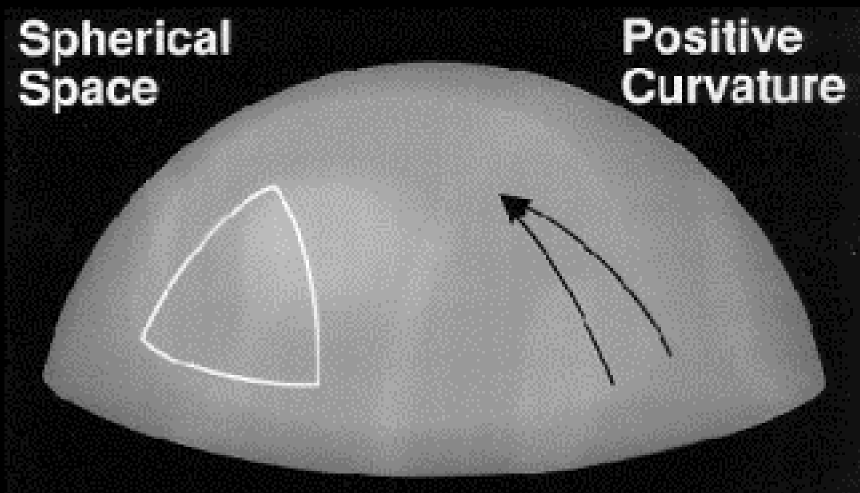
# Robertson Walker metric

local spacetime line element

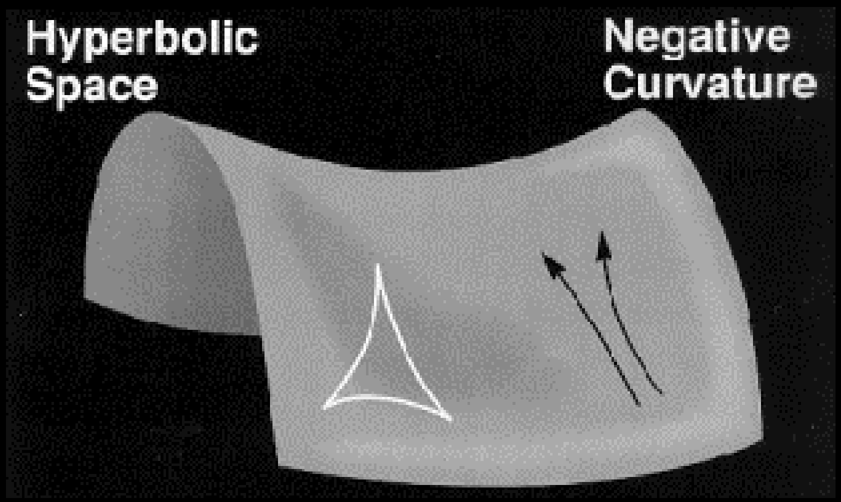
$$\mathrm{d}s^2 = \mathrm{d}(ct)^2 - R^2\left[\frac{\mathrm{d}\chi^2}{1-k\chi^2} + \chi^2[\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2]\right]$$

$$r = R\chi \quad (R = \mathrm{const})$$

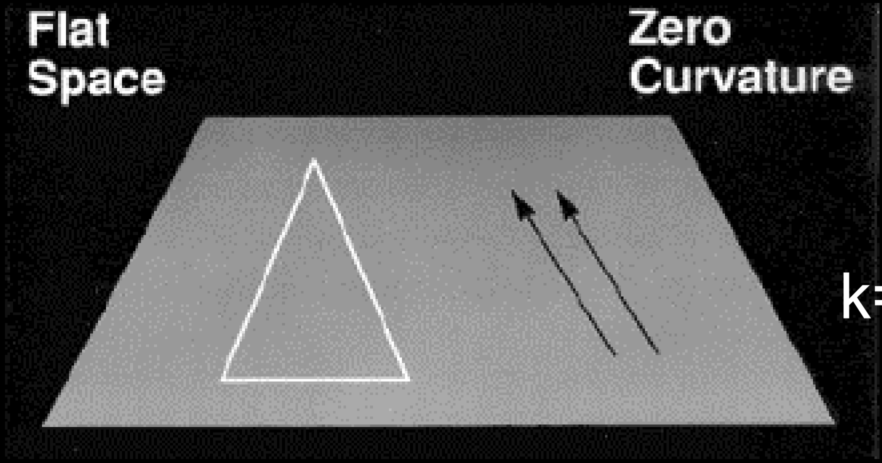
# 2-D Examples of Curved Spaces




k=+1

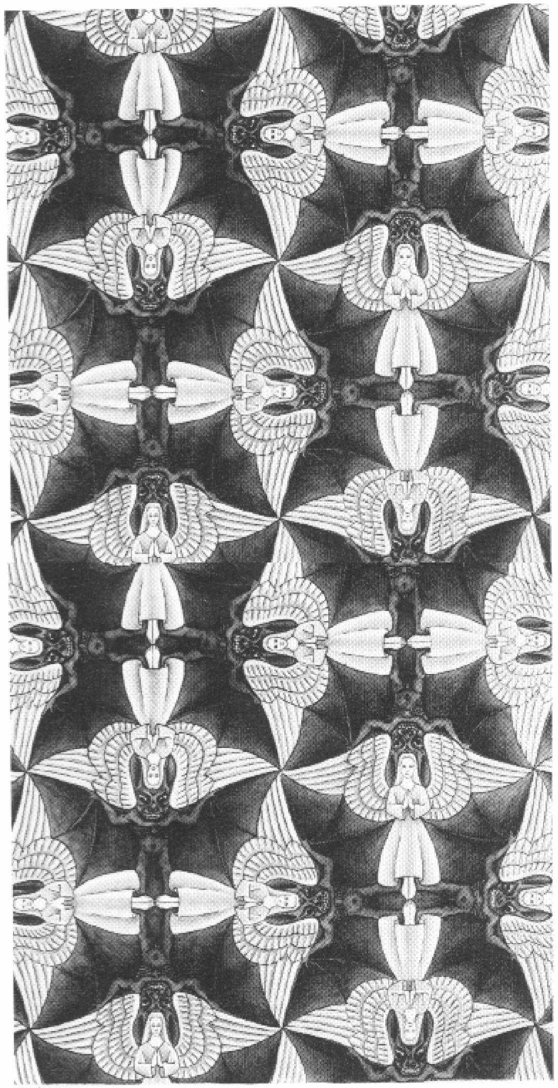



k=-1

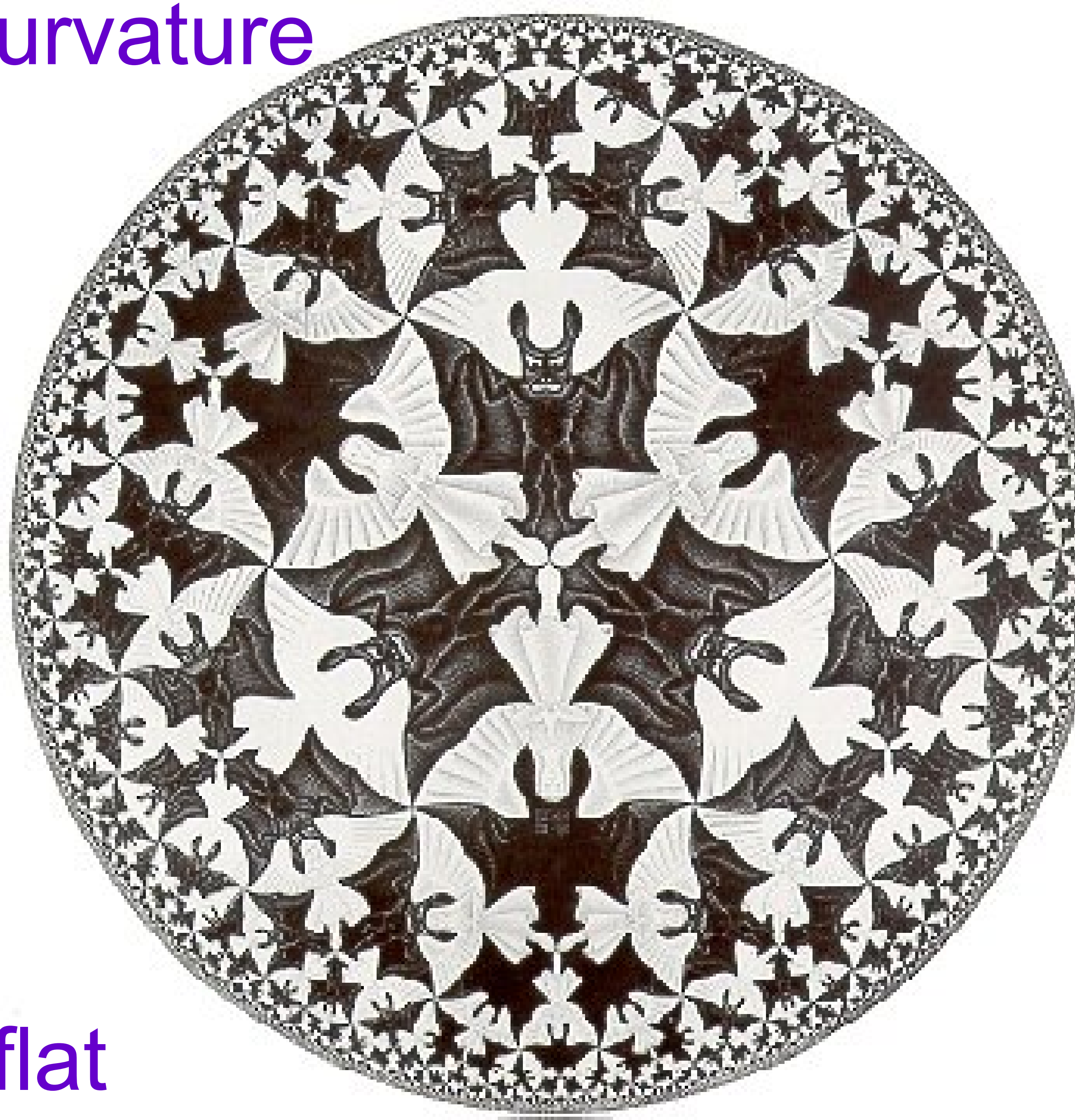



k=0

# negative curvature



flat

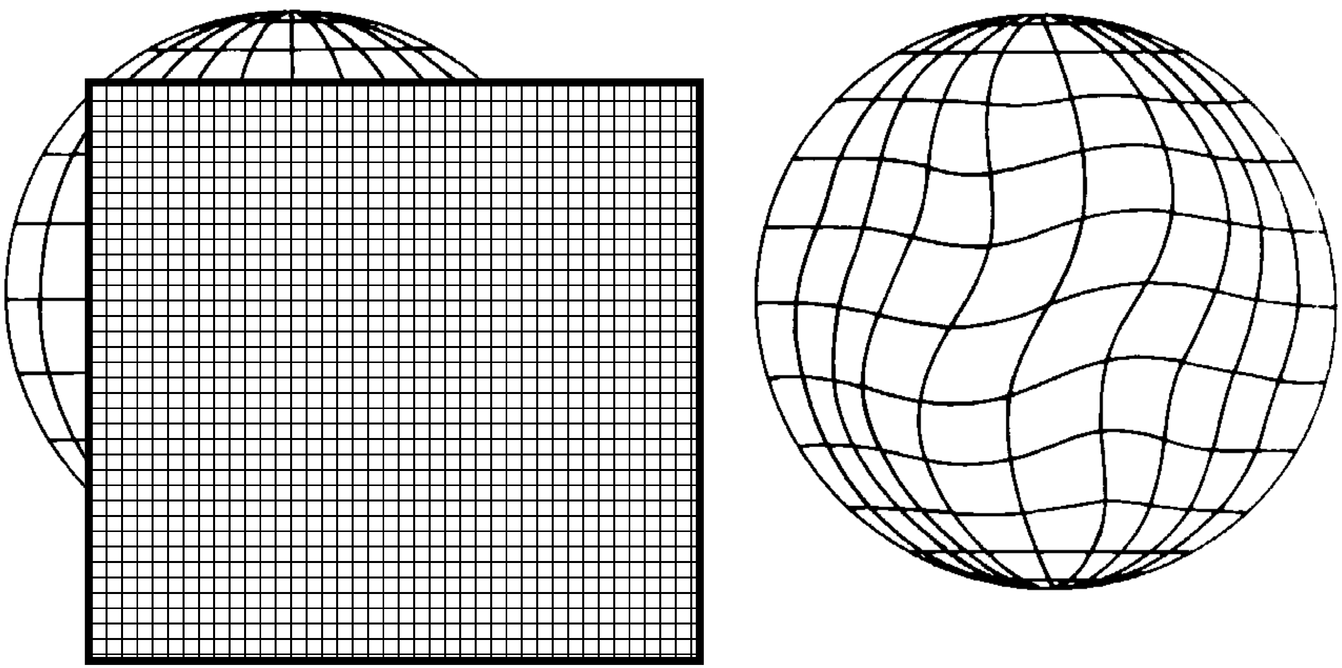


# General Relativity: a Local Gauge Theory

general metrics $$x \cdot y = \sum_{\nu=0}^{3} \sum_{\mu=0}^{3} g_{\mu\nu} x^{\mu} y^{\nu}$$

$$s \cdot s = \sum_{\nu=0}^{3} \sum_{\mu=0}^{3} g_{\mu\nu} s^{\mu} s^{\nu}$$

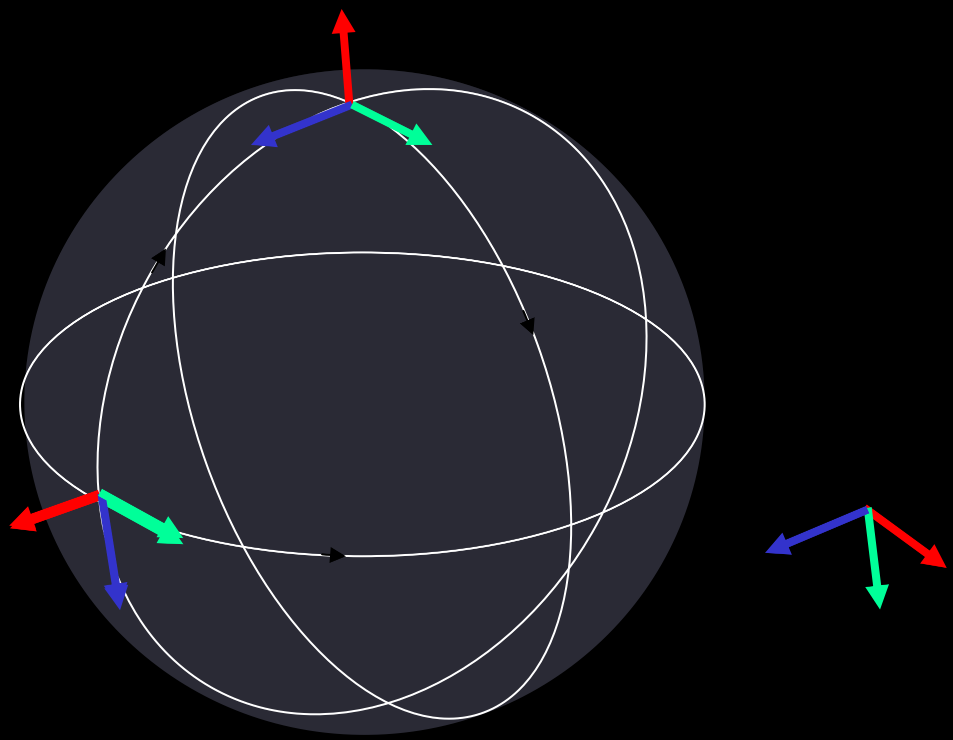
local orthonormal $$\mathrm{d}s^2 = (c\,\mathrm{d}\tau)^2 = g_{00}(c\,\mathrm{d}t)^2 + g_{11}\,\mathrm{d}x^2 + g_{22}\,\mathrm{d}y^2 + g_{33}\,\mathrm{d}z^2$$



a b

# Problem: Move local orthonormal coordinates

described by Riemann tensor



# The Friedmann-Lemaitre equations

$$\left(\frac{\dot{R}(t)}{R(t)}\right)^2 = \tfrac{8}{3}\pi G\rho(t) - \frac{kc^2}{R(t)^2}, \qquad k = E/m$$

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)$$

# The relativistic Friedmann Equation

$$\left(\frac{\dot{R}(t)}{R(t)}\right)^2 = H(t)^2 = \tfrac{8}{3}\pi G\rho(t) - \frac{kc^2}{R(t)^2}$$

- Replace mass density by energy density $\rho c^2$
- contributions from photons etc. as well as matter

- curvature
  - $k$ = +1: positive curvature
  - $k$ = 0: flat
  - $k$ = -1: negative curvature

# Scaled Density Ω

$$\left(\frac{\dot{R}(t)}{R(t)}\right)^2 = H(t)^2 = \tfrac{8}{3}\pi G\rho(t) - \frac{kc^2}{R(t)^2}$$

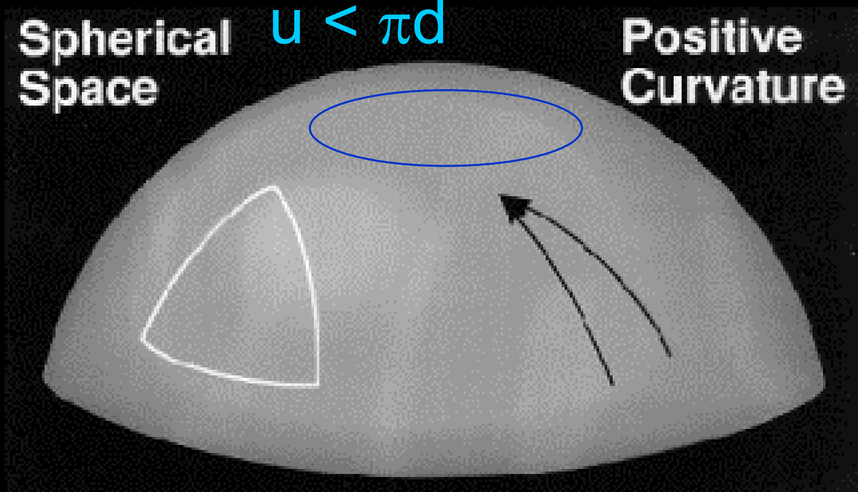
- Energy density for $k$=0
  - critical density
    $\rho_c = 3H^2 / 8\pi G$
    $= 1.88 \cdot 10^{-29}\ g/cm^3\ (H^2/100\ km/s/Mpc)$
  - define $\Omega \equiv \rho(t) / \rho_c(t)$
  - then

$$H(t)^2(1-\Omega) = -\frac{kc^2}{R(t)^2}$$

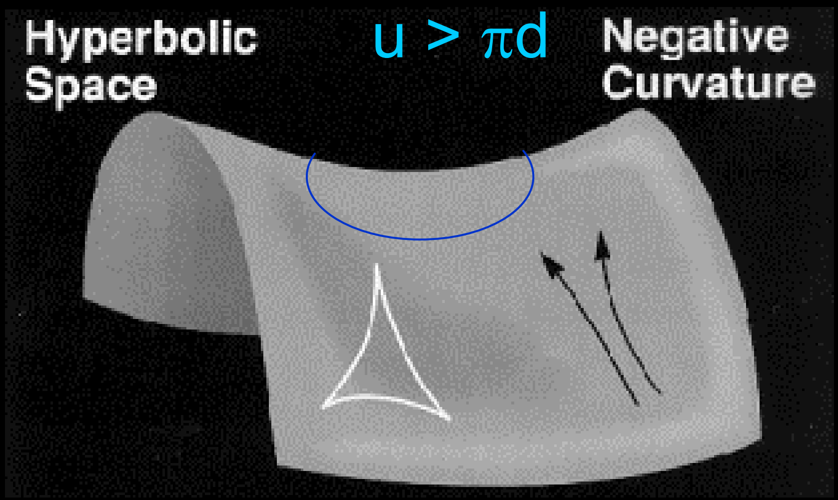
- $H^2$, $R^2$, $c^2$ all positive
  - $\Omega = 1 \leftrightarrow k = 0$
  - $\Omega < 1 \leftrightarrow k < 0$
  - $\Omega > 1 \leftrightarrow k > 0$

# 2-D Examples of Curved Spaces

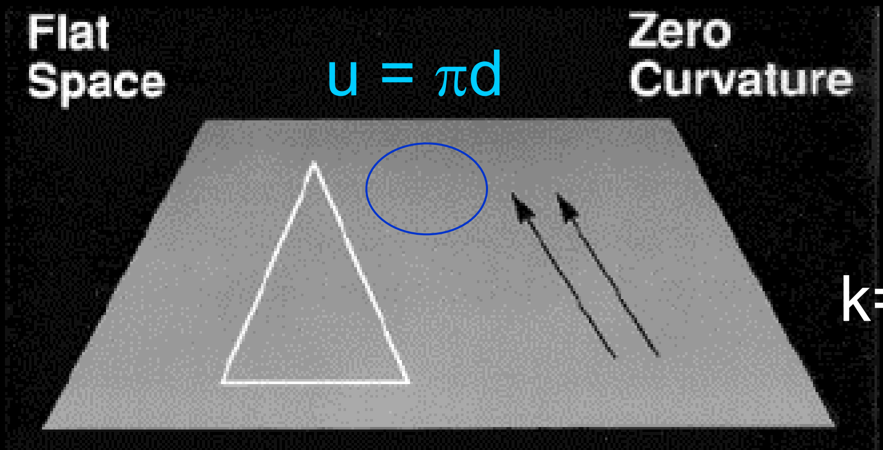
circumference of circle




k=+1

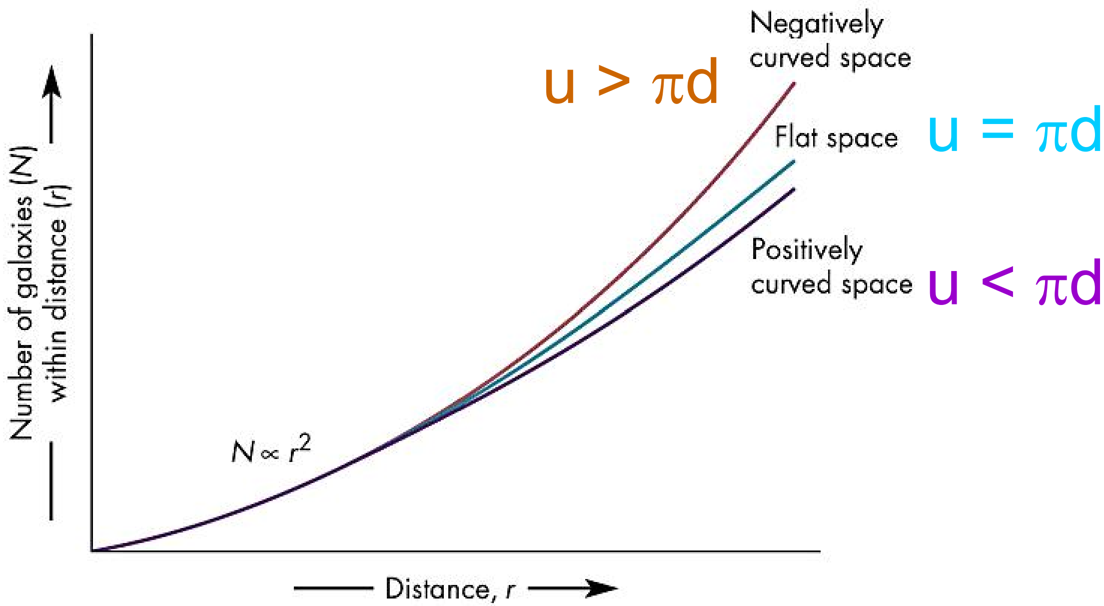


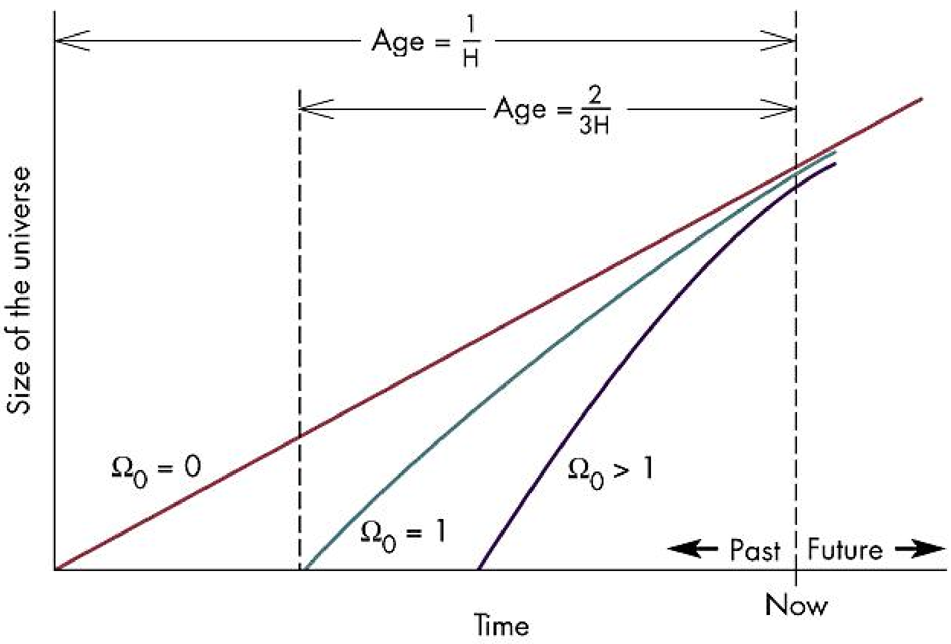

k=-1

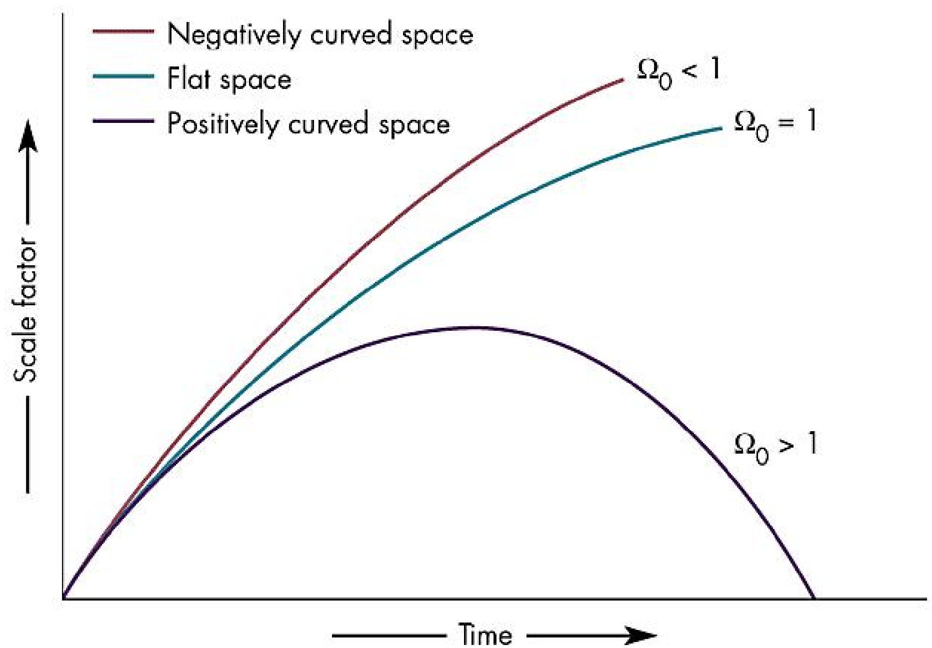



k=0

# Curvature and History



# The Cosmological Constant Λ

$$\left(\frac{\dot{R}(t)}{R(t)}\right)^2 = H(t)^2 = \tfrac{8}{3}\pi G\rho(t) - \frac{kc^2}{R(t)^2} + \frac{\Lambda}{3}$$

- Friedmann equation (like Newton) not static
  - Einstein believed Universe is static (pre-Hubble)
  - introduced cosmological constant Λ to allow this
    - basically an integration constant in Einstein's equations
    - can also be expressed as $\Omega_\Lambda = \Lambda/3H^2$; then usually call density parameter $\Omega_m$ to distinguish the two

$$H(t)^2\left(1-\Omega_m-\Omega_\Lambda\right) = -\frac{kc^2}{R(t)^2}$$

# The equation of state

- We have introduced the pressure $P$
  - need to relate $P$ and $\rho$
    - this relation depends on substance (equation of state)
  - some useful equations of state:
    - non-relativistic gas: $P = nk_BT = \rho k_BT/\mu$ ($\mu$=particle mass) since $3k_BT = \mu\langle v^2\rangle$, we have $P/\rho c^2 = \langle v^2\rangle/3c^2 << 1$, i.e. $P_m \approx 0$
    - radiation (ultra-relativistic): $P/\rho c^2 = ⅓$ so $P_r = ⅓\,\rho c^2$
    - Λ: as its energy density is constant with time, $P_\Lambda = -\rho c^2$ this gives acceleration, since $\rho c^2 + 3P < 0$

# Radiation pressure

$$P = \frac{\mathrm{d}\,N_\gamma}{\mathrm{d}\,A \cdot \mathrm{d}\,t} \cdot 2\frac{E_\gamma}{c}\cos\theta = \frac{F}{c}\cdot 2\cos\theta$$

Reflection

radiation onto area:
*P* = pressure
E/c = momentum of γ
*F* = flux

to one side

$$F = \frac{1}{2}\frac{\mathrm{d}\,E}{\mathrm{d}\,V}\cdot c\cos\theta$$

$$P = \frac{\mathrm{d}\,E}{\mathrm{d}\,V}\int_0^1 \cos^2\theta\,\mathrm{d}\cos\theta$$

$$= \frac{1}{3}\frac{\mathrm{d}\,E}{\mathrm{d}\,V}$$

(it is **not** radiation pressure acting here)

# The fluid equation revisited

$$\dot{\rho} + 3\frac{\dot{R}}{R}\left(\rho + \frac{P}{c^2}\right) = 0$$

- Radiation

  $P_r/c^2 = ⅓\, \rho_r \longrightarrow \frac{\dot{\rho}}{\rho} = -4\frac{\dot{R}}{R}$

  $\rho_r \propto R^{-4}$

- cold matter

  $P_m \approx 0 \longrightarrow \frac{\dot{\rho}}{\rho} = -3\frac{\dot{R}}{R}$

  $\rho_m \propto R^{-3}$

- $\Lambda$

  $P_\Lambda = -\rho_\Lambda c^2 \longrightarrow \dot{\rho} = 0$

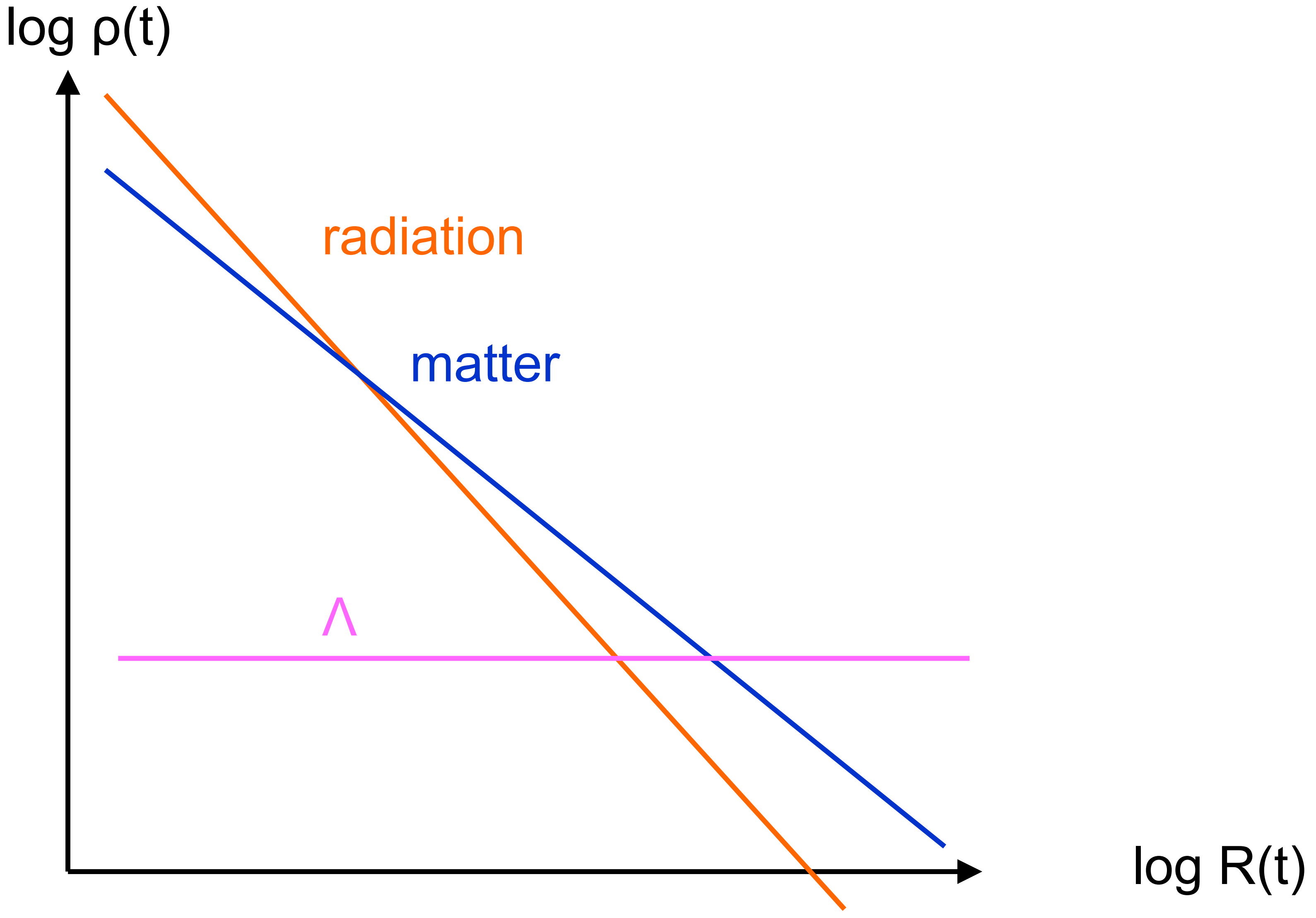
  $\rho_\Lambda$ = constant

- More general form

  $P = w\rho c^2 \longrightarrow \frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{R}}{R}$

  $\rho \propto R^{-3(1+w)}$

# History of densities



# Cosmological models

$$\left(\frac{\dot{R}(t)}{R(t)}\right)^2 = \tfrac{8}{3}\pi G\rho(t) - \frac{kc^2}{R(t)^2}$$

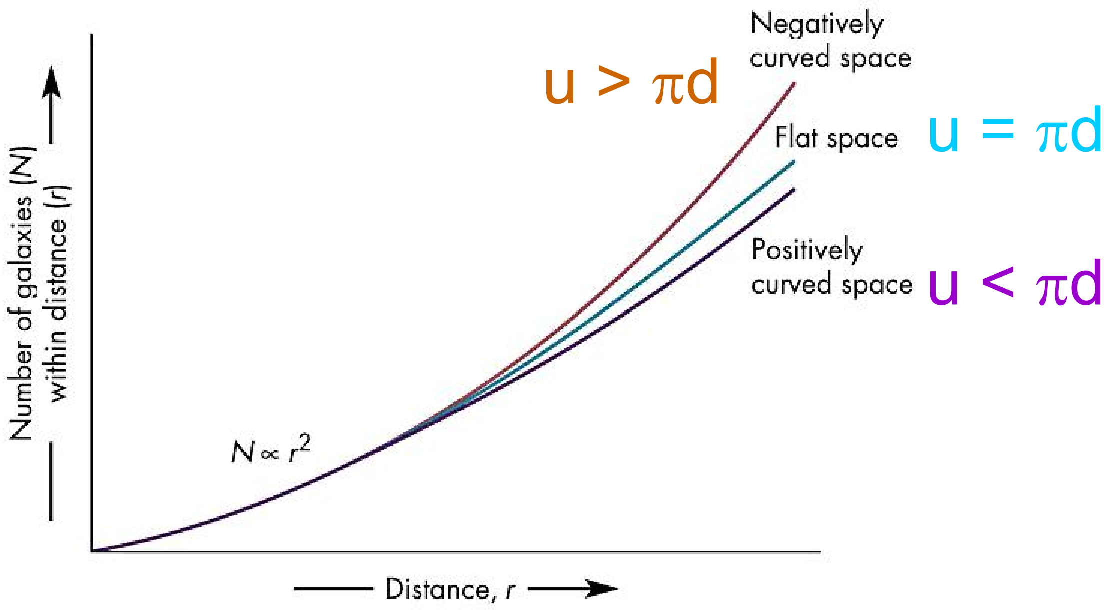
Can now substitute (using $a = R/R_0$)

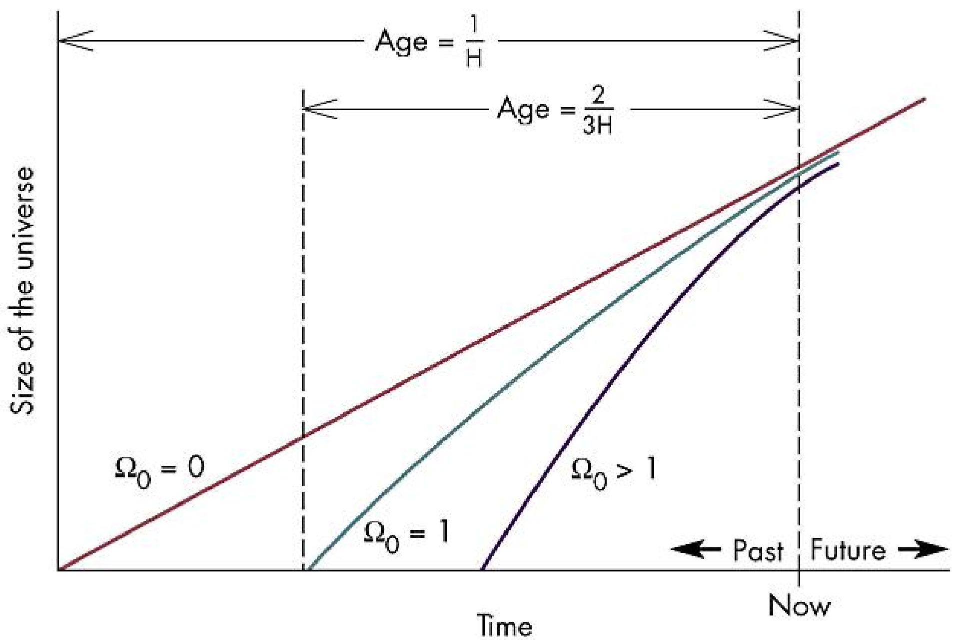
- $\rho = \rho_0 a^{-3}$ (matter),
- $\rho = \rho_0 a^{-4}$ (radiation), or
- $\rho = \rho_0$ ($\Lambda$)

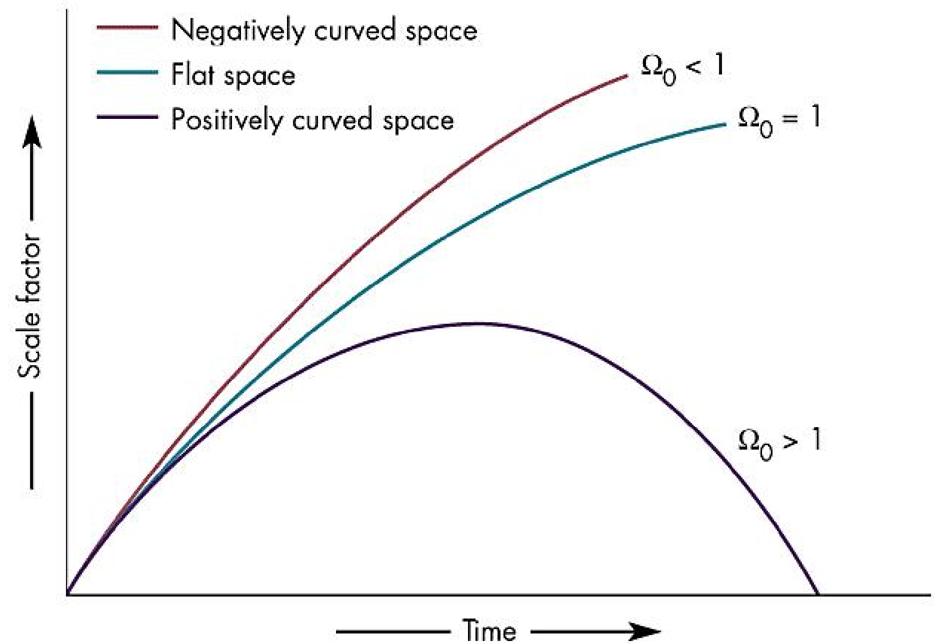
***in real life a combination of all three!***

to get a differential equation for $R(t)$

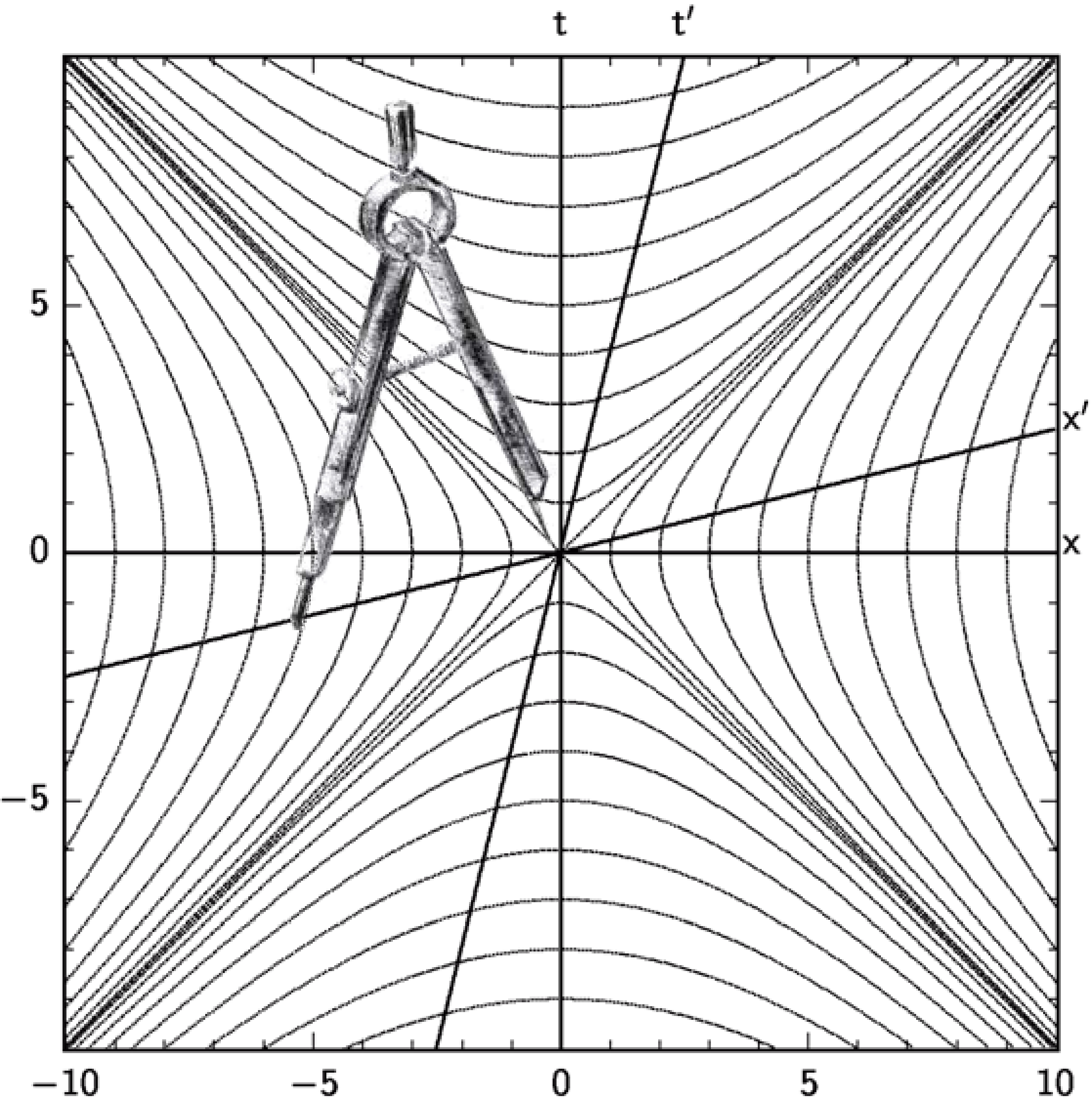
# Curvature and History



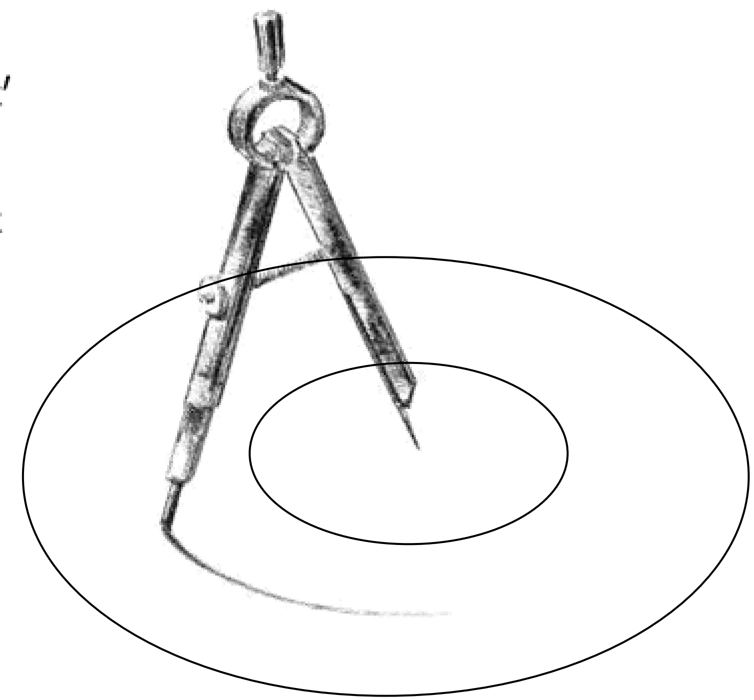



# Special Relativity: Minkowski diagram




hyperbolae replace circles to measure space or time distances



# The Beginning of Time

Everything has a start!

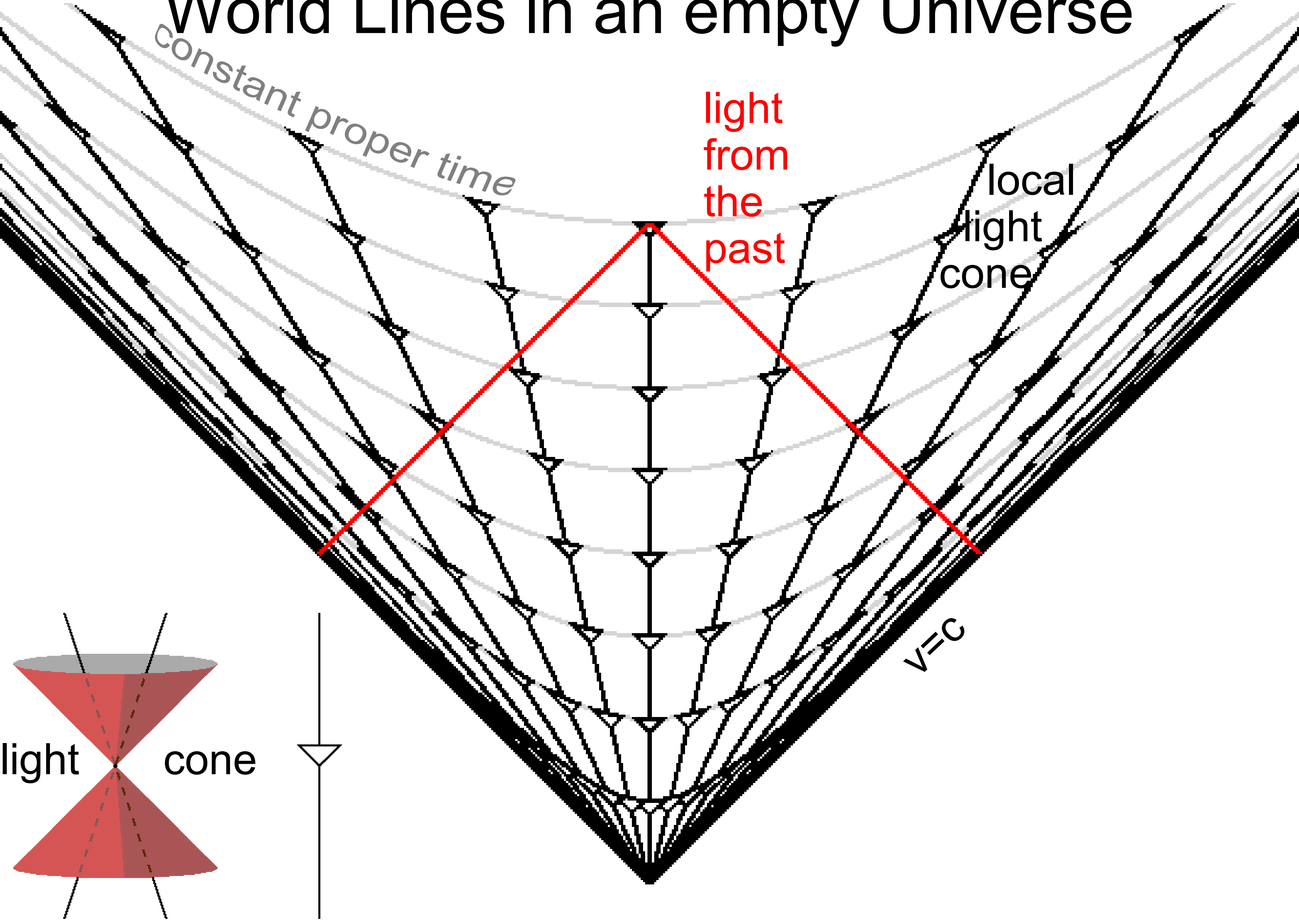
lowest temperature: 0 K = -273,15 °C

lowest elevation: Earth's Centre at 6 378 140 m **below** sea level

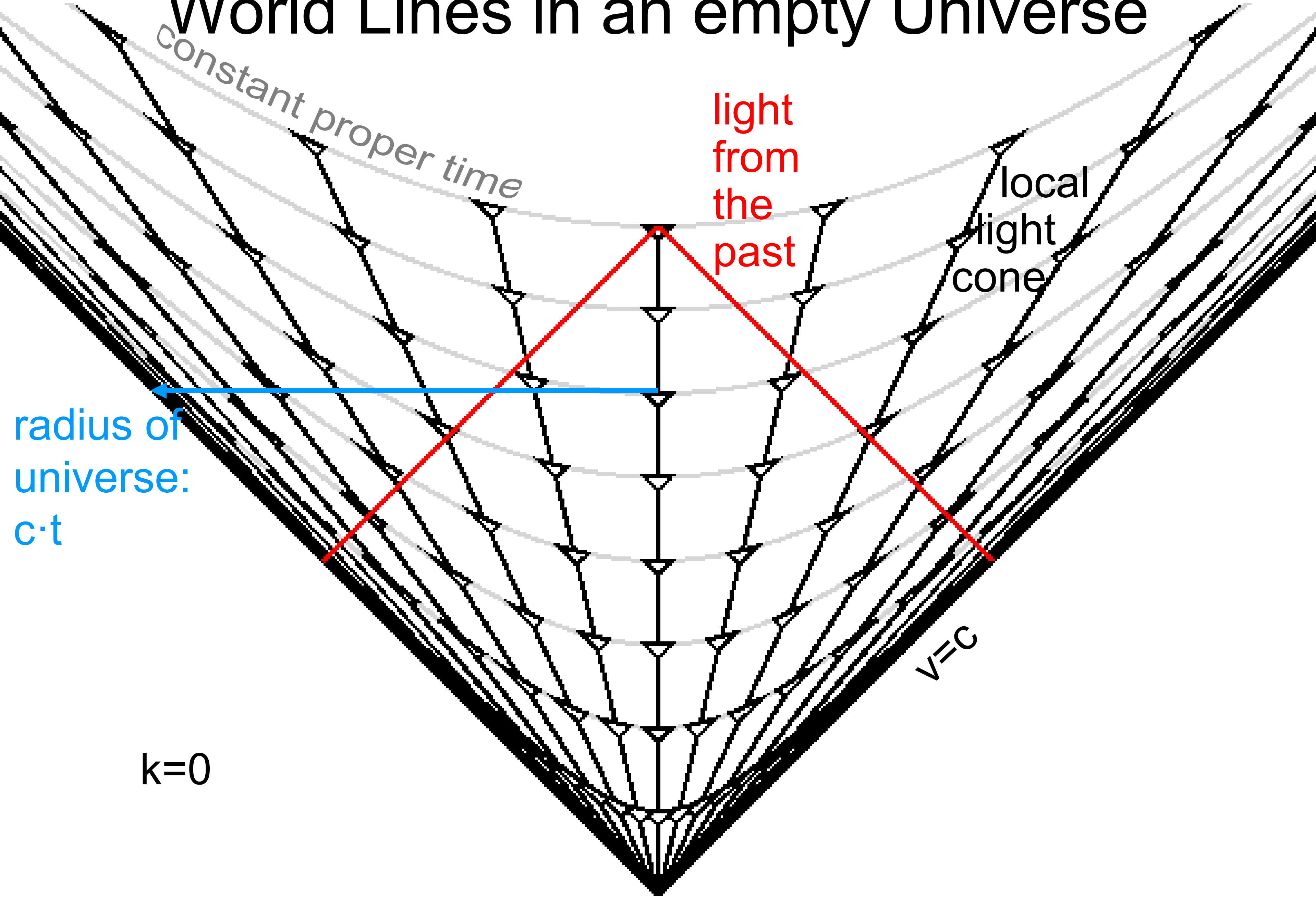
absolute starting point of every way north: the South Pole

absolute starting point of time: the Big Bang

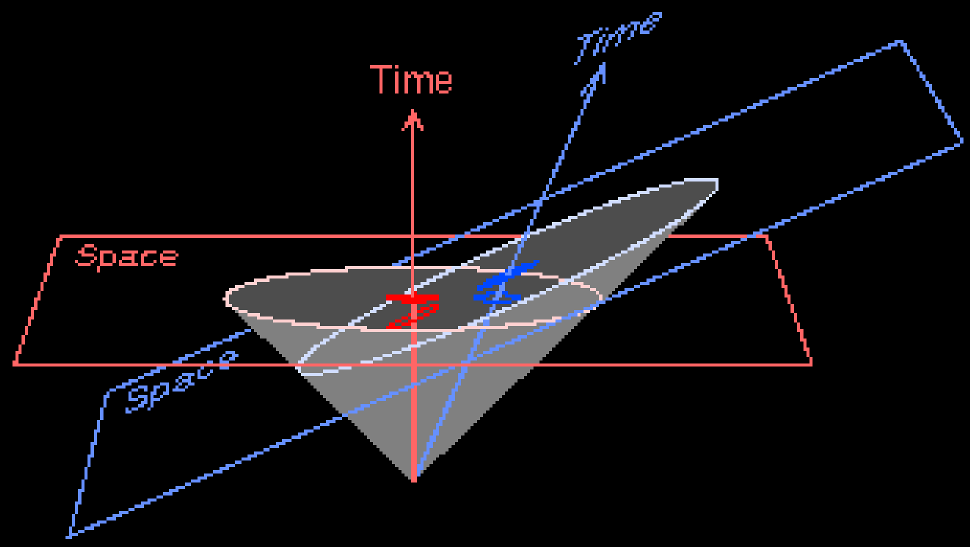
# World Lines in an empty Universe



# World Lines in an empty Universe



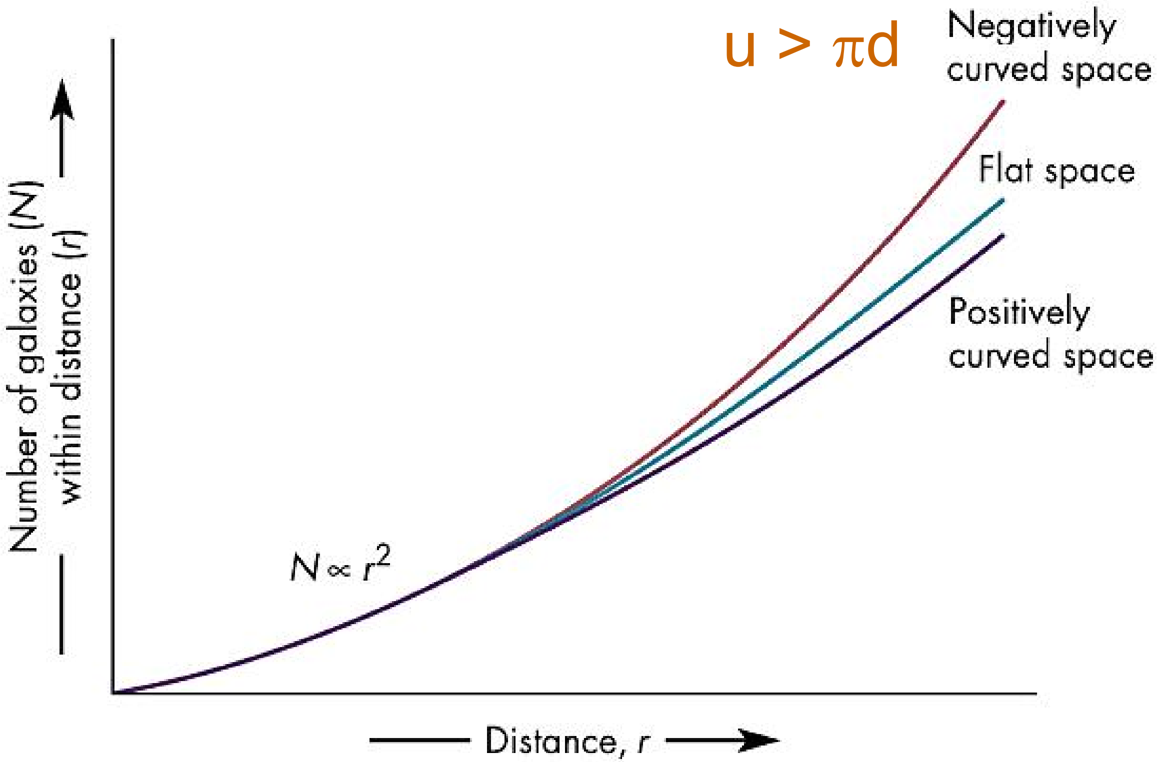
# The Lightcone




planes
of constant time
for two inertial systems

# Curvature and History

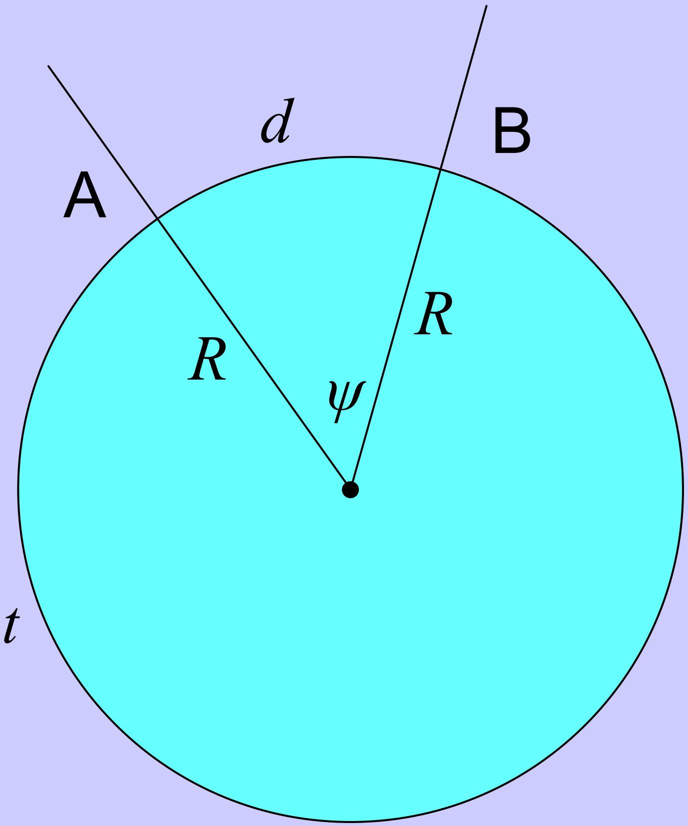
applies also to brightness




define distance:

- along constant local time ($t = \text{const}$) ??
- along constant proper time ($\tau = \text{const}$)
- by angle: arc b / distance r = angle φ
- by luminosity / flux ratio

# Expansion of Sphere




radius $R = \mathrm{v}_R \cdot t$

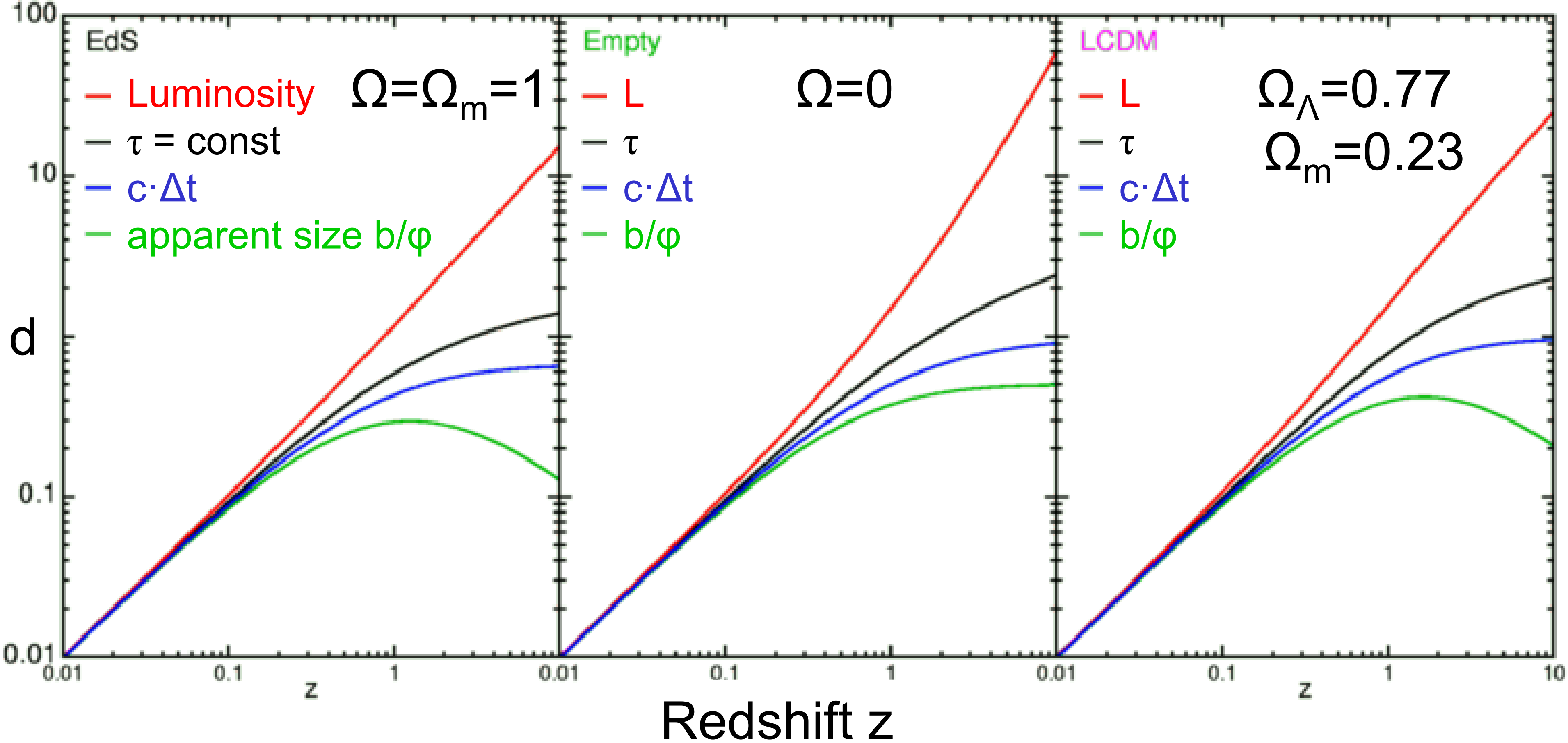
distance on surface $d = \psi \cdot R = \psi \cdot \mathrm{v}_R \cdot t = \mathrm{v}_d \cdot t$

$$\frac{\mathrm{v}_d}{d} = \frac{\mathrm{v}_R}{R} = \frac{1}{t} = H$$

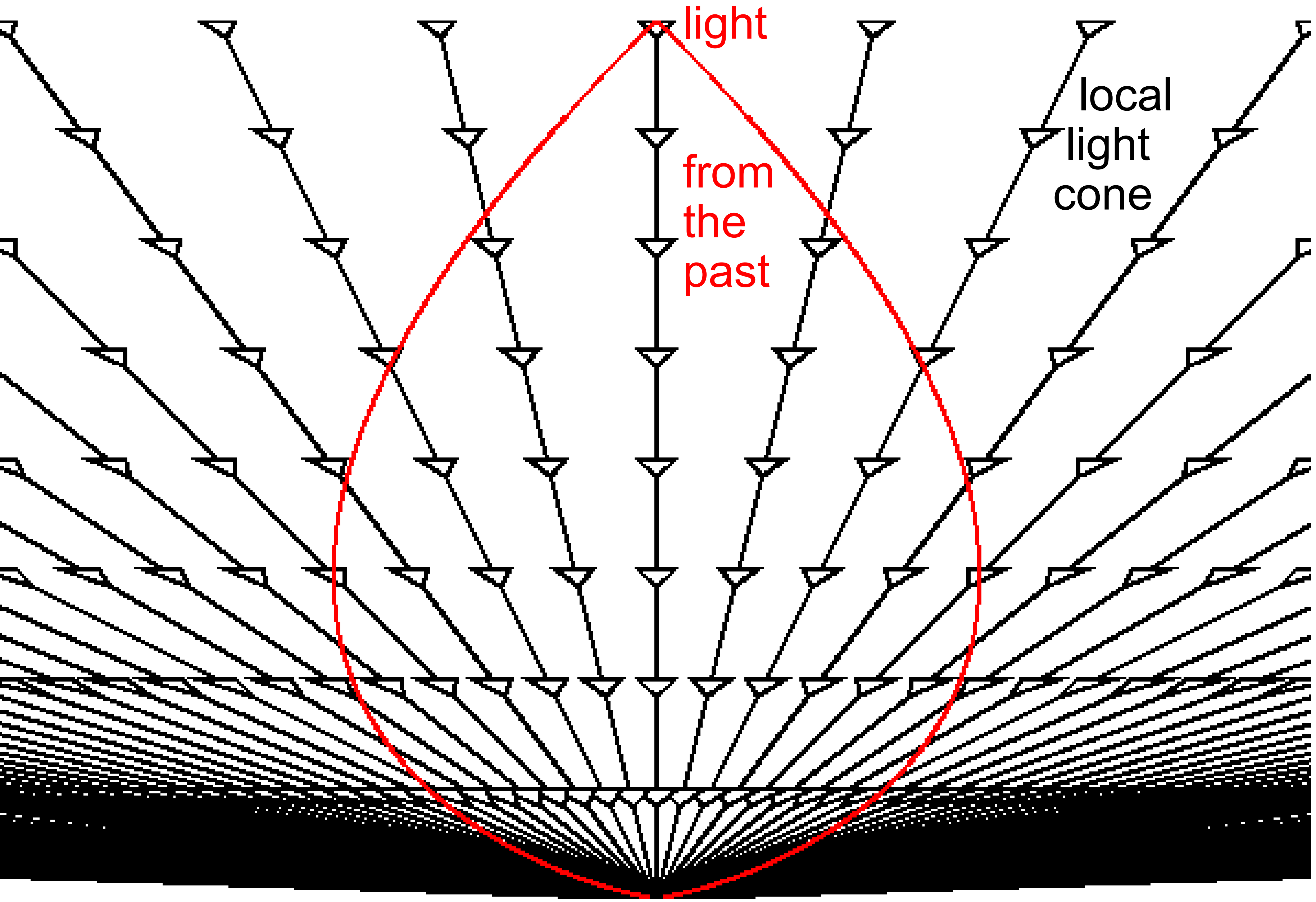
for large distances: $d = d_\tau$ is along constant *proper time* $\tau$

$\mathrm{v}_d > c$ possible

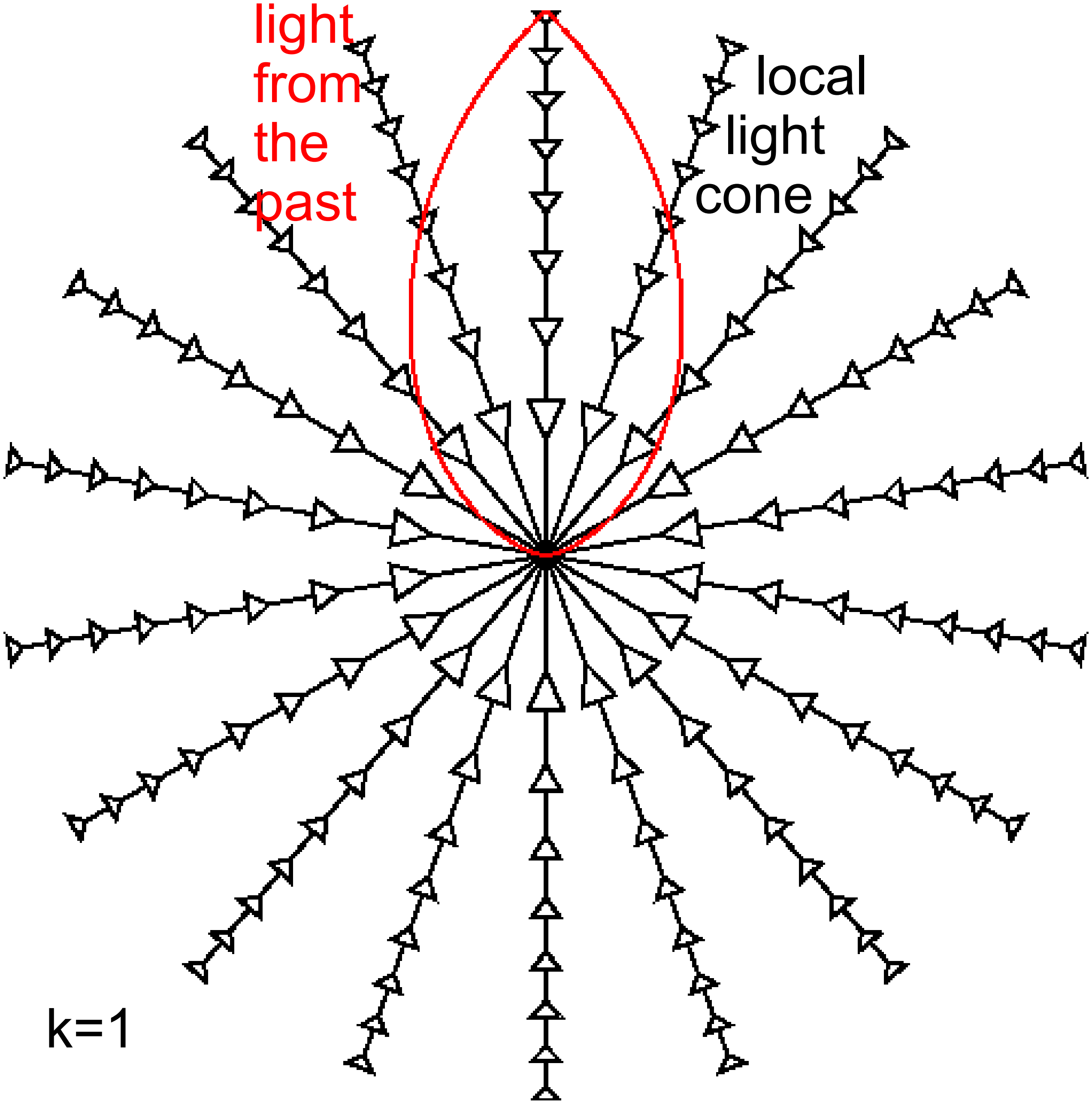
# The physical *distance*



# World Lines in empty Universe



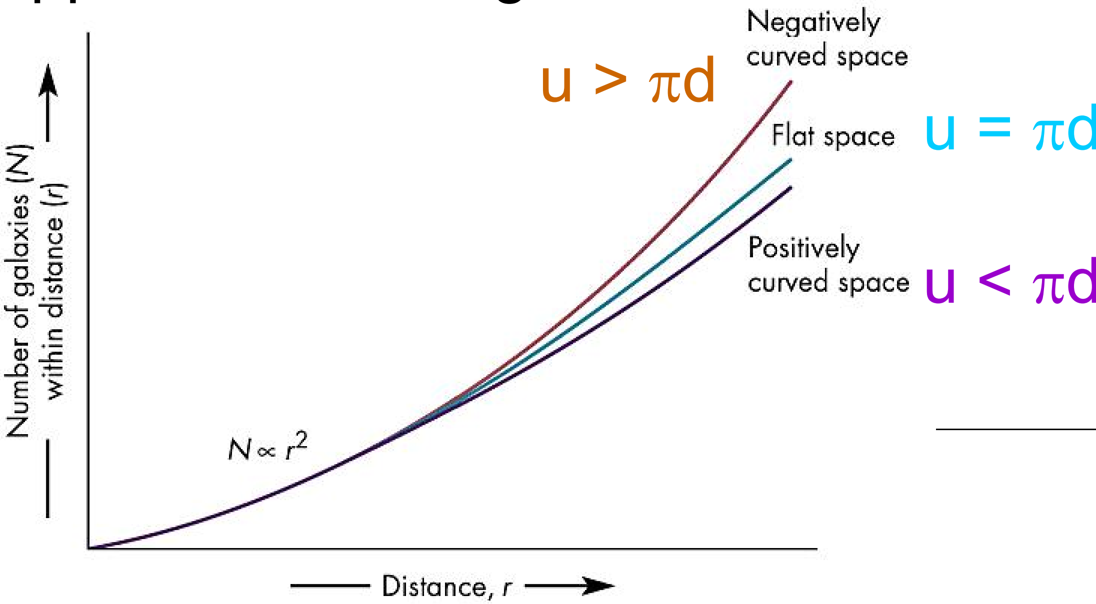
# World Lines in a curved Universe

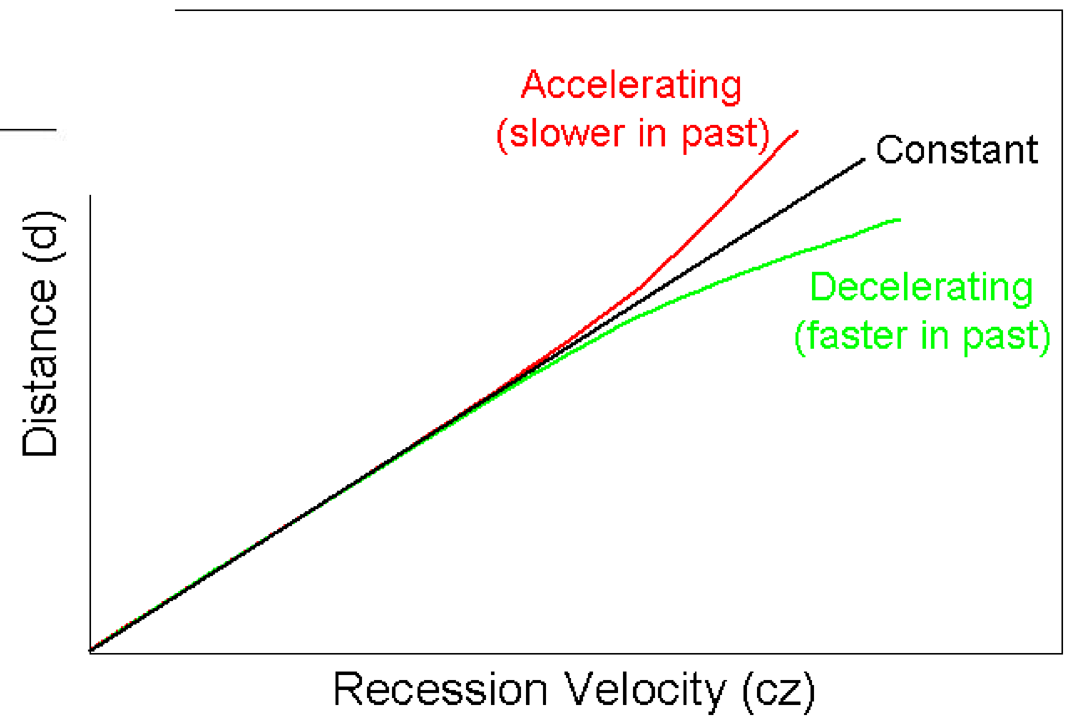


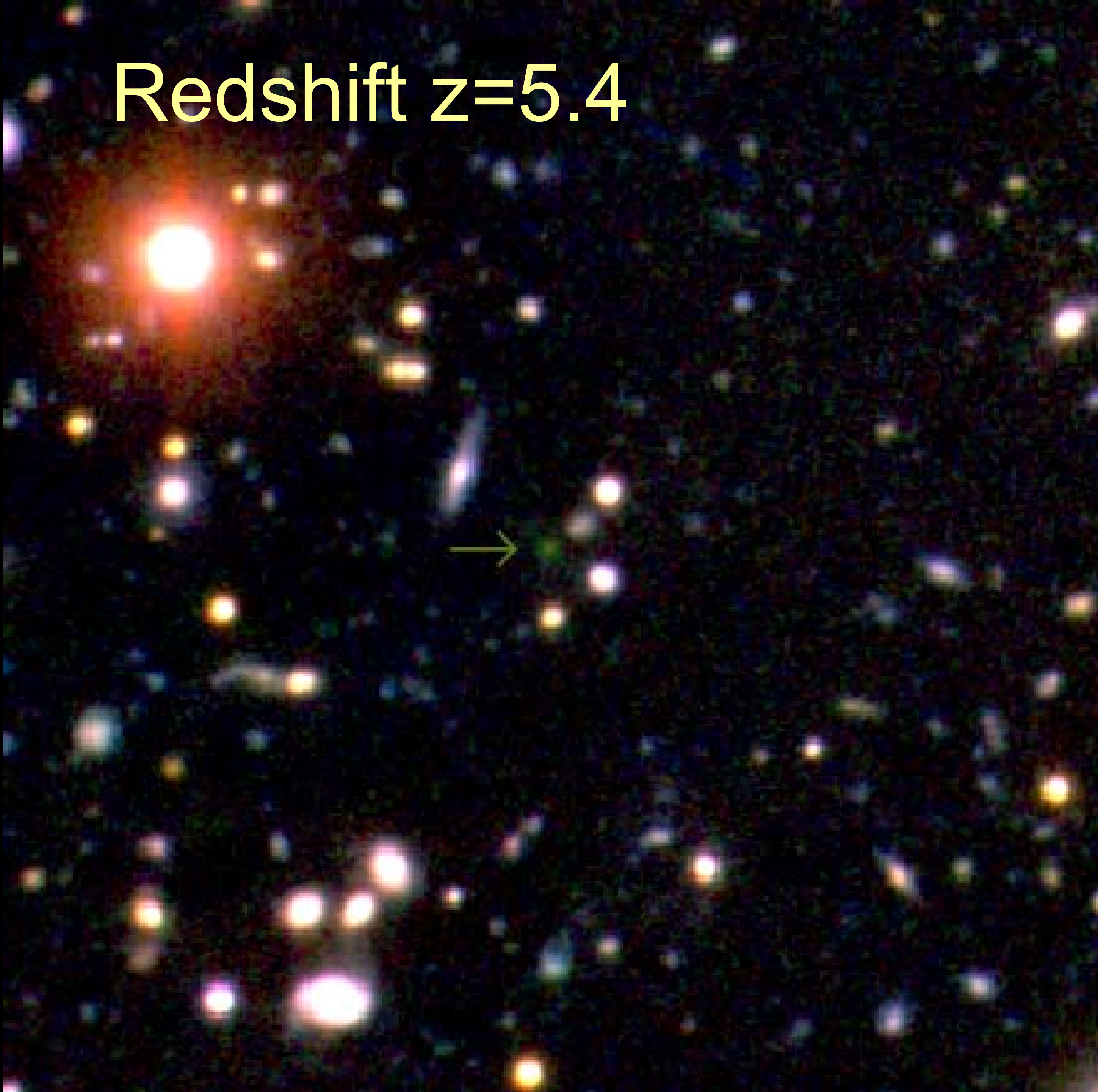

we can see
at most half of this
Universe

# Curvature and History

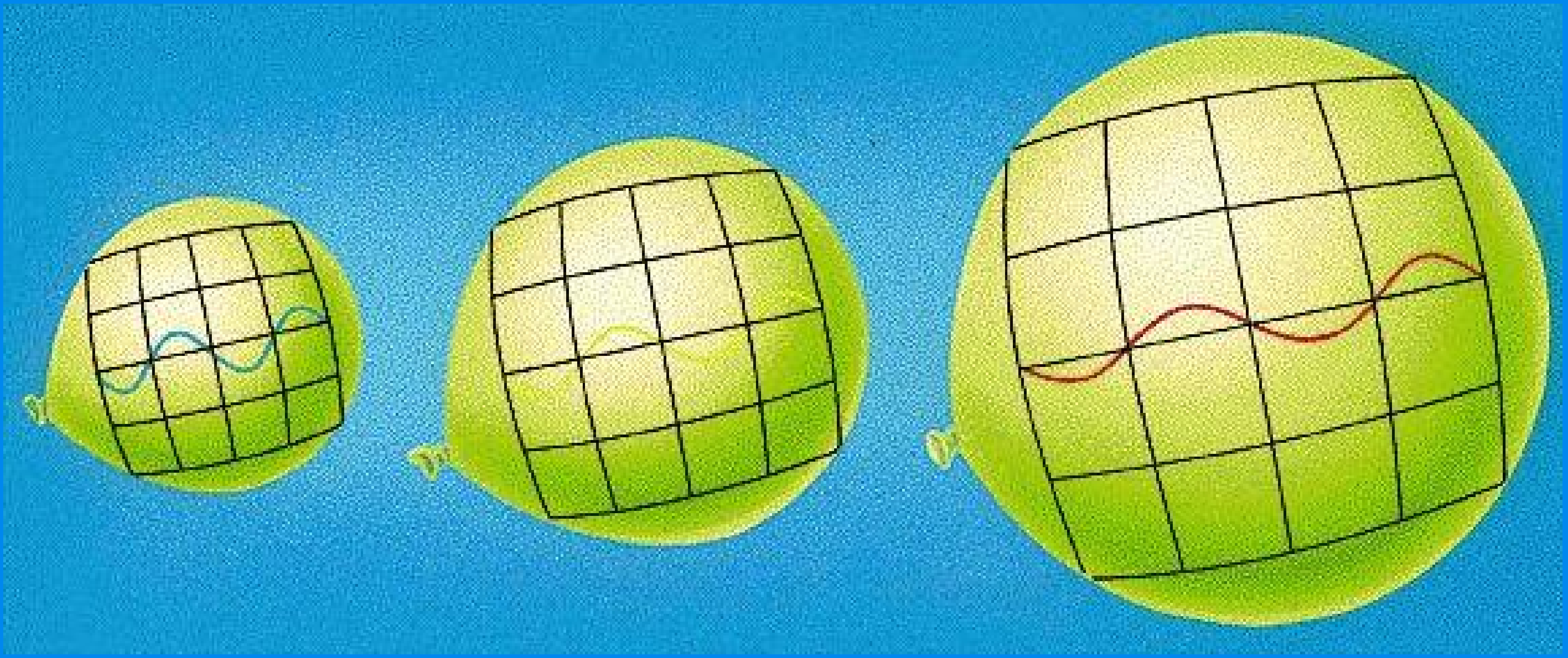
applies also to brightness



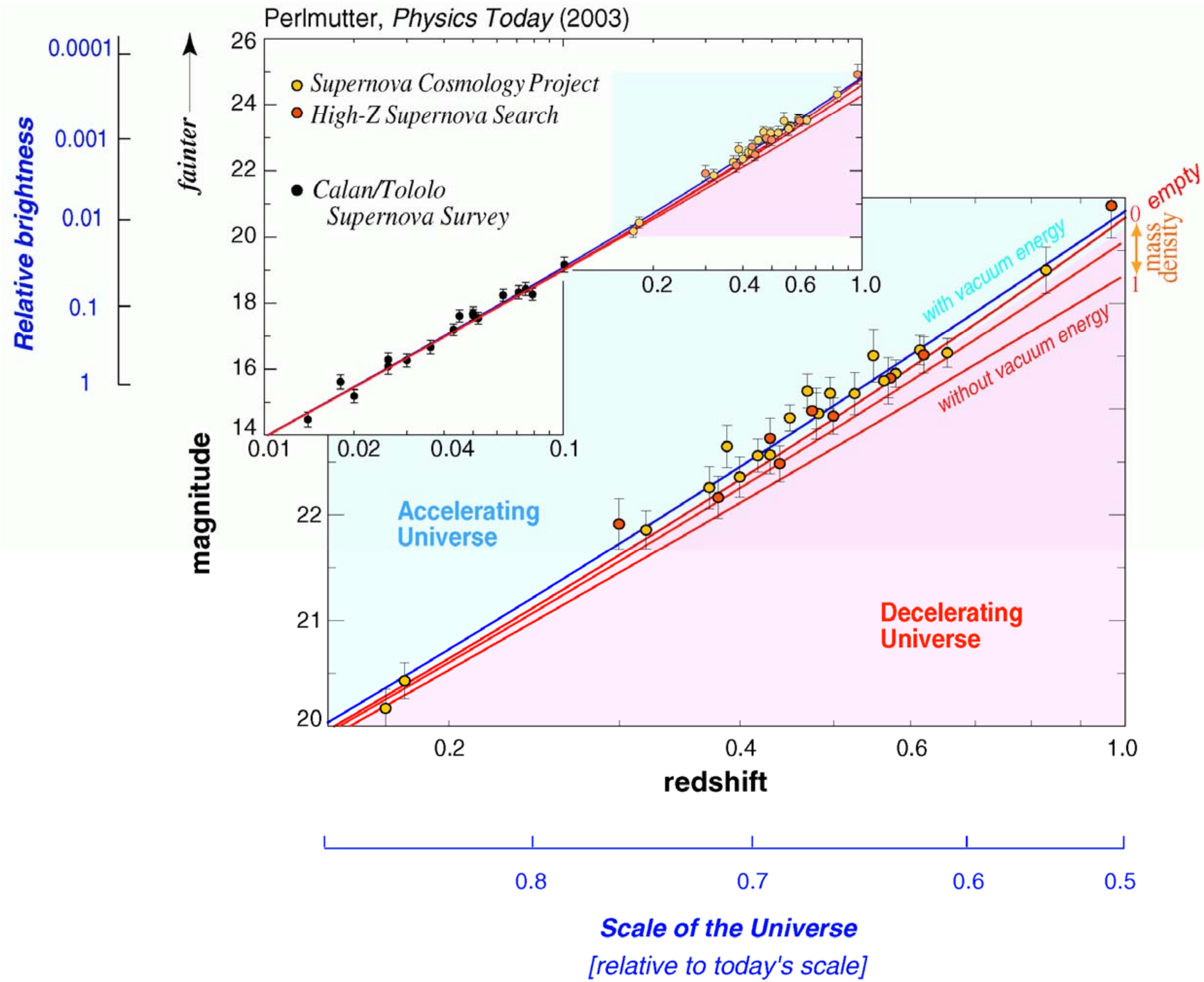

Redshift z=5.4

# Doppler Effekt from Expansion

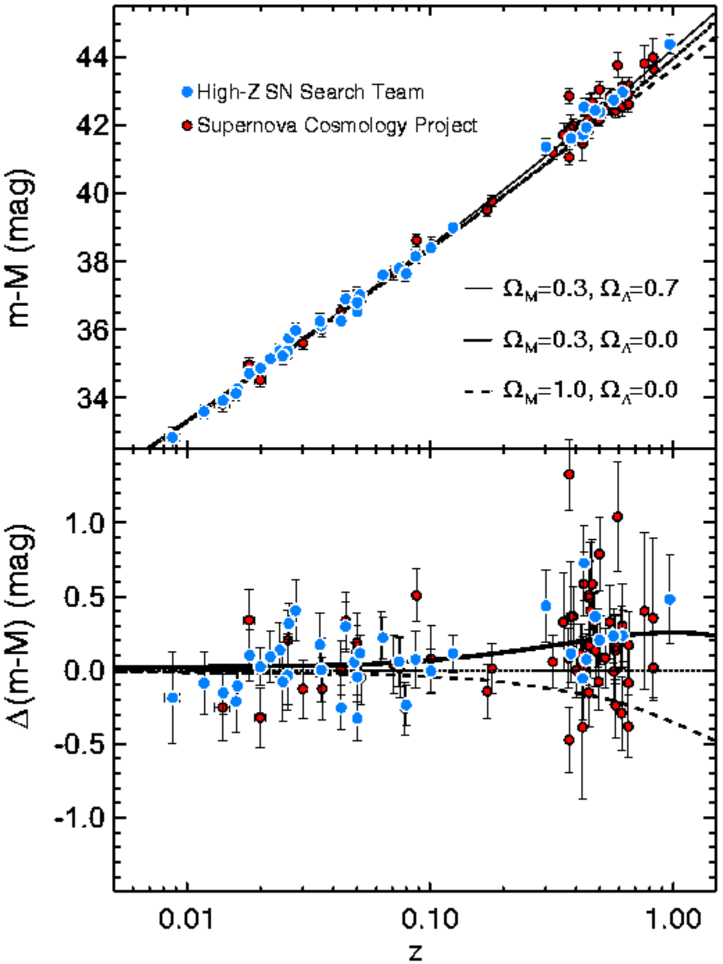



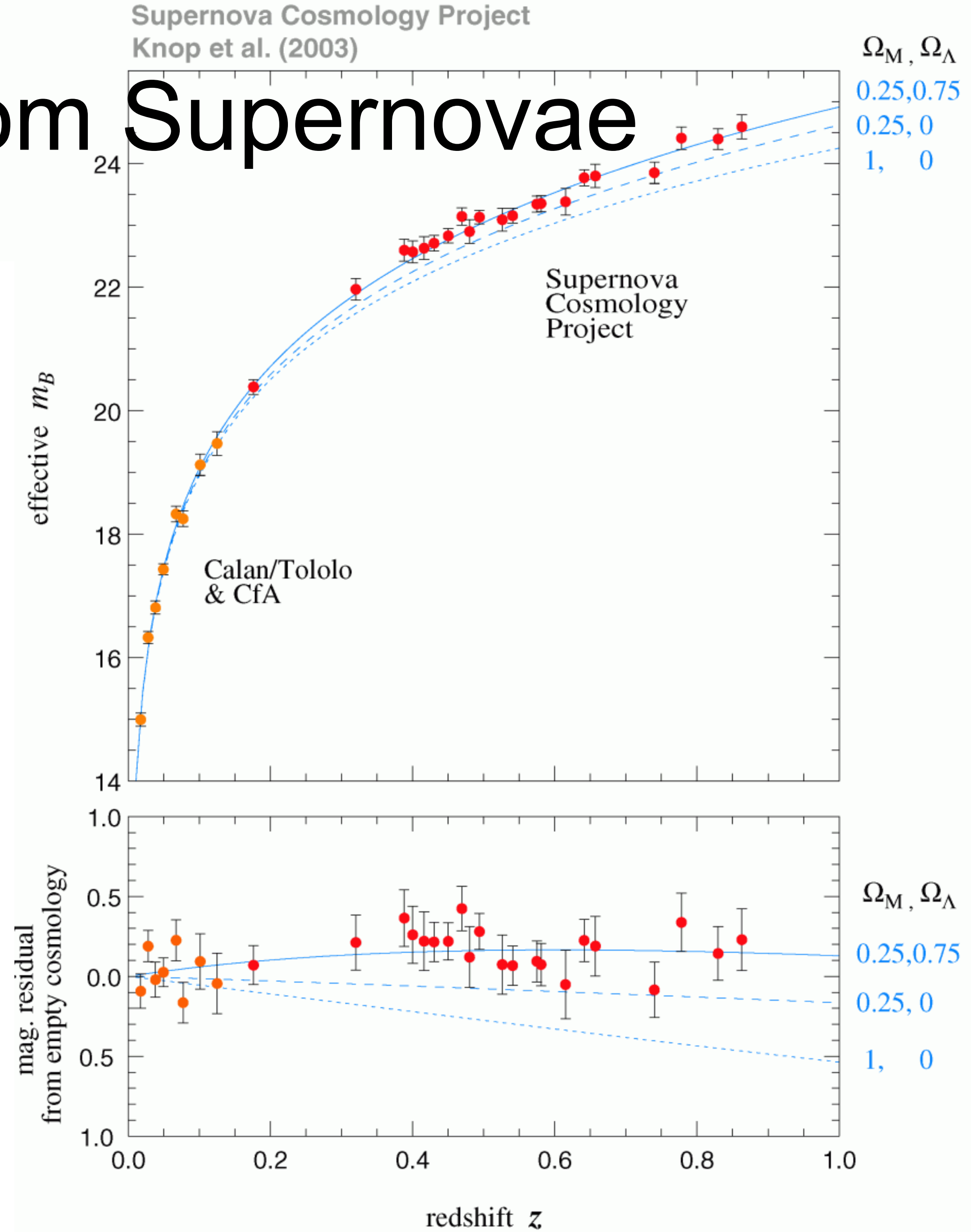
$$1 + z = \frac{\lambda'}{\lambda} = \frac{R'}{R}$$

# Distance Scale: Supernovae Type Ia

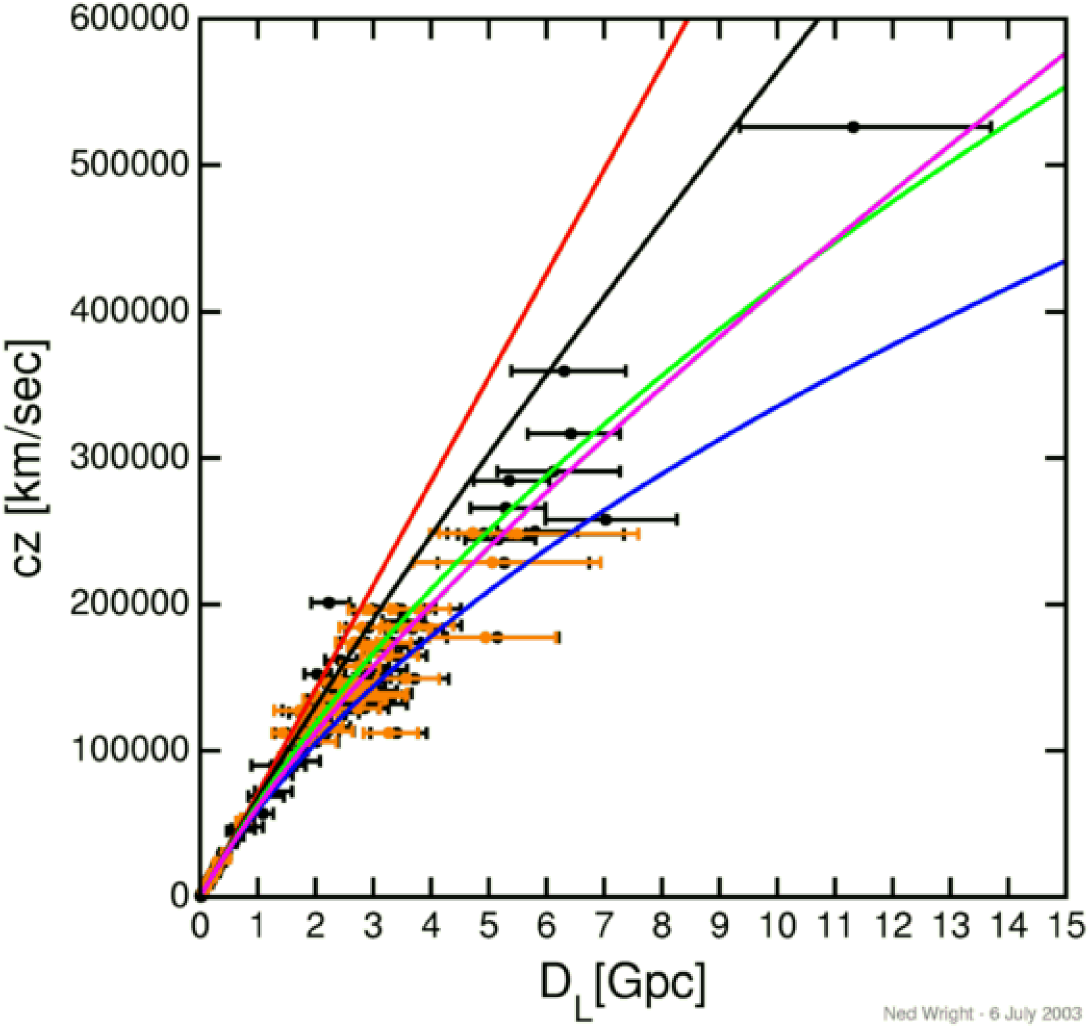


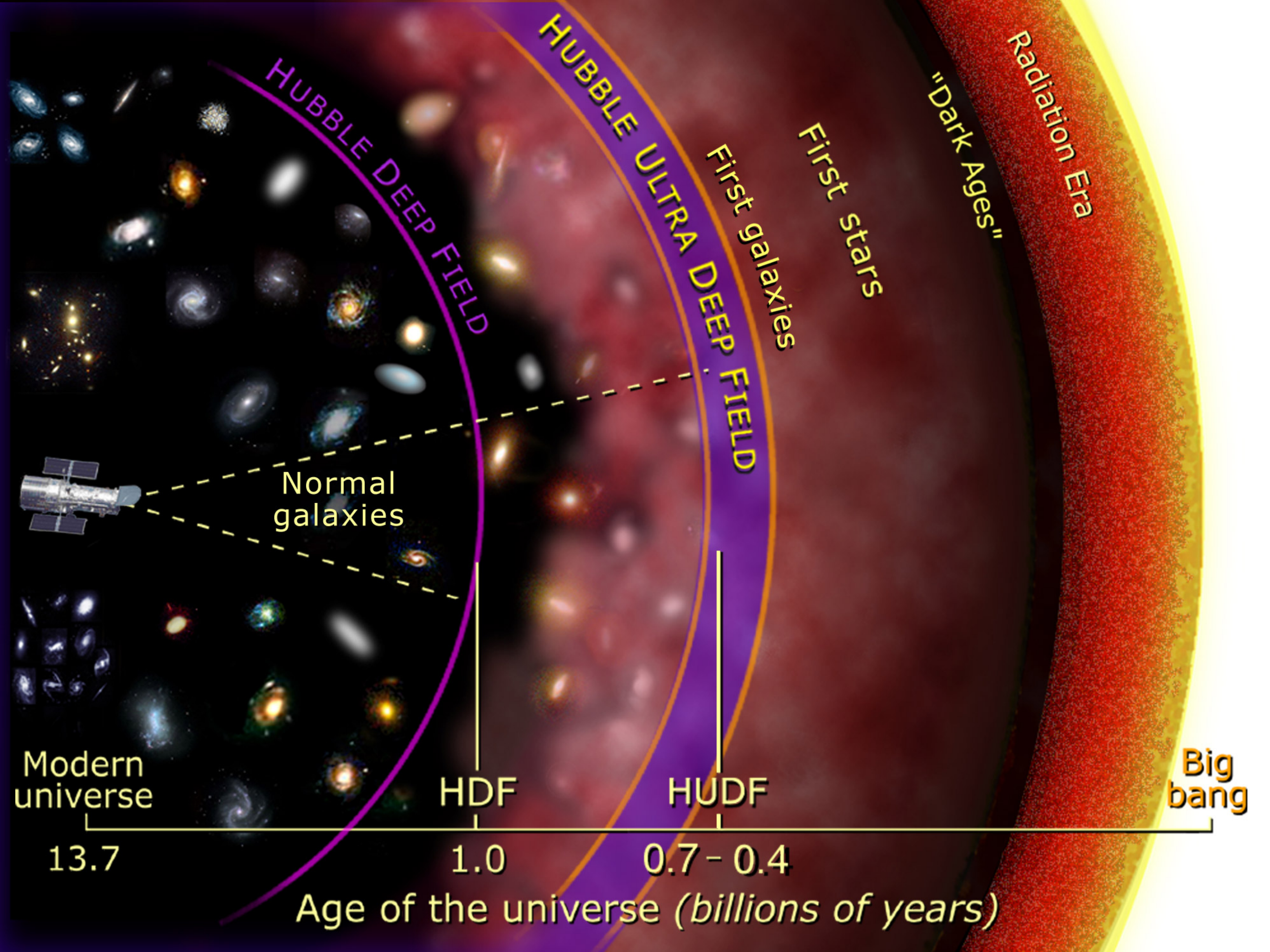
# Hubble Plots from Supernovae



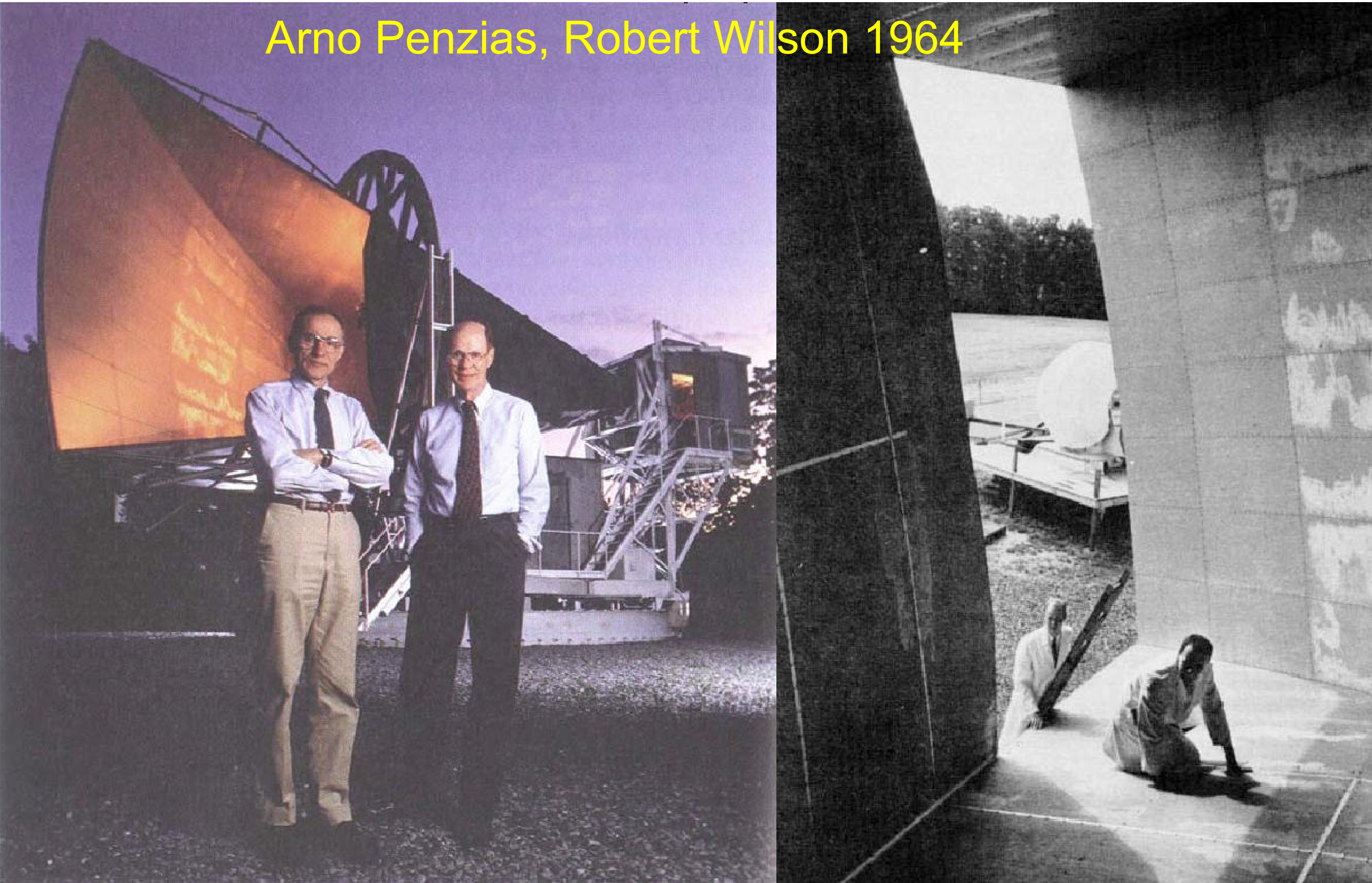
# Hubble Plots from Supernovae



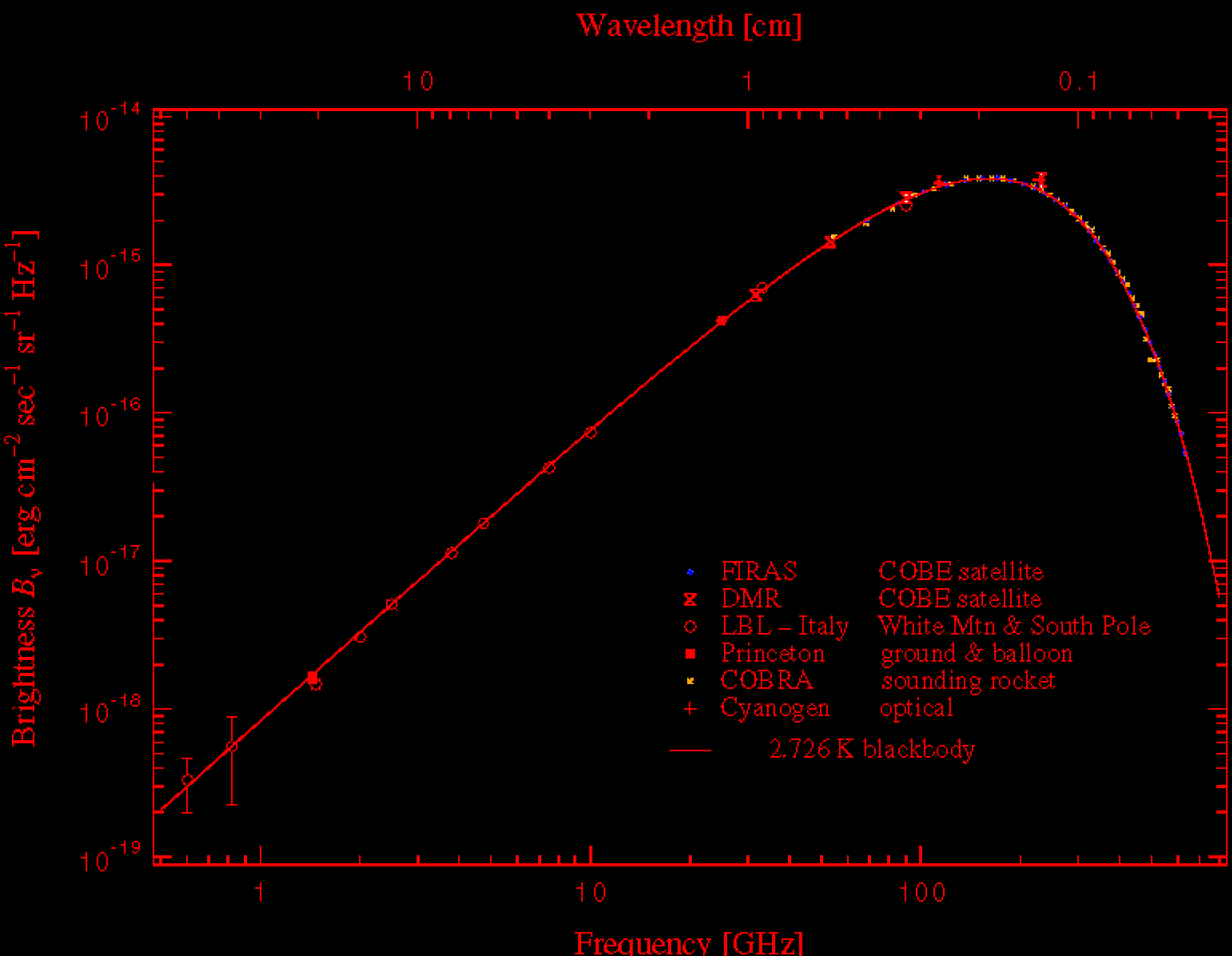


HUBBLE DEEP FIELD
HUBBLE ULTRA DEEP FIELD
First galaxies
First stars
"Dark Ages"
Radiation Era
Normal galaxies
Modern universe
HDF
HUDF
Big bang
13.7
1.0
0.7 – 0.4
Age of the universe (billions of years)

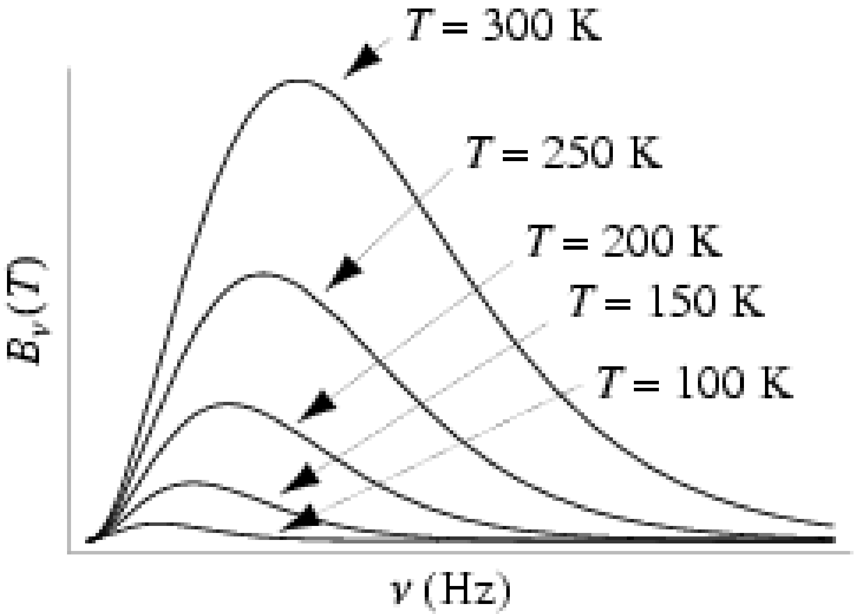
# Cosmic Microwave Background

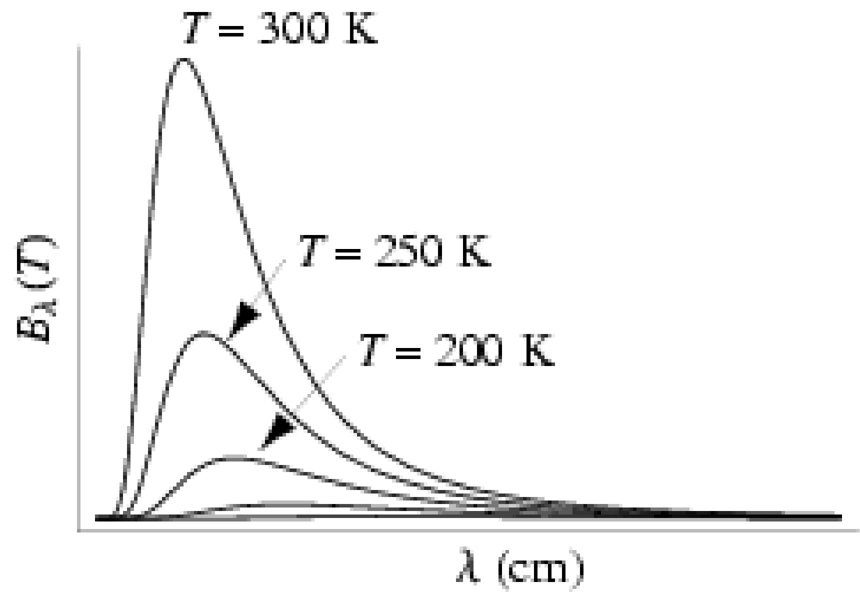


# Temperature of the Universe



# Planck's Radiators (Black Body)



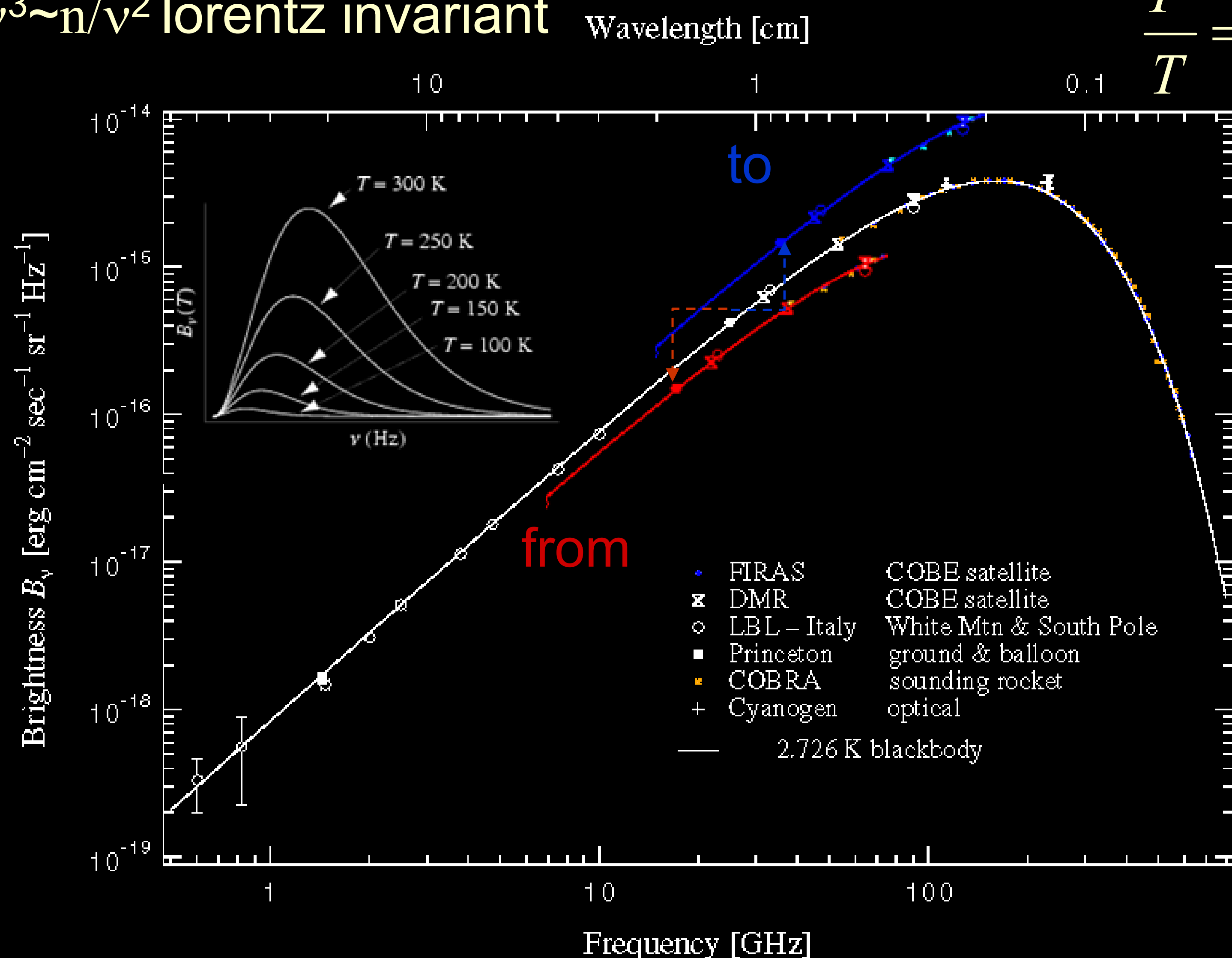

$$B_\nu = \frac{\mathrm{d}^4 E}{\mathrm{d}t \;\; \mathrm{d}A \cos\vartheta \;\; \mathrm{d}\Omega \;\; \mathrm{d}\nu} = \frac{2h}{c^2} \frac{\nu^3}{e^{h\nu/kT} - 1}$$

Lorentz transformation $B_\nu \rightarrow B'_{\nu'}$ with $\dfrac{T'}{T} = \dfrac{\nu'}{\nu}$

# Dipole: Doppler Shift

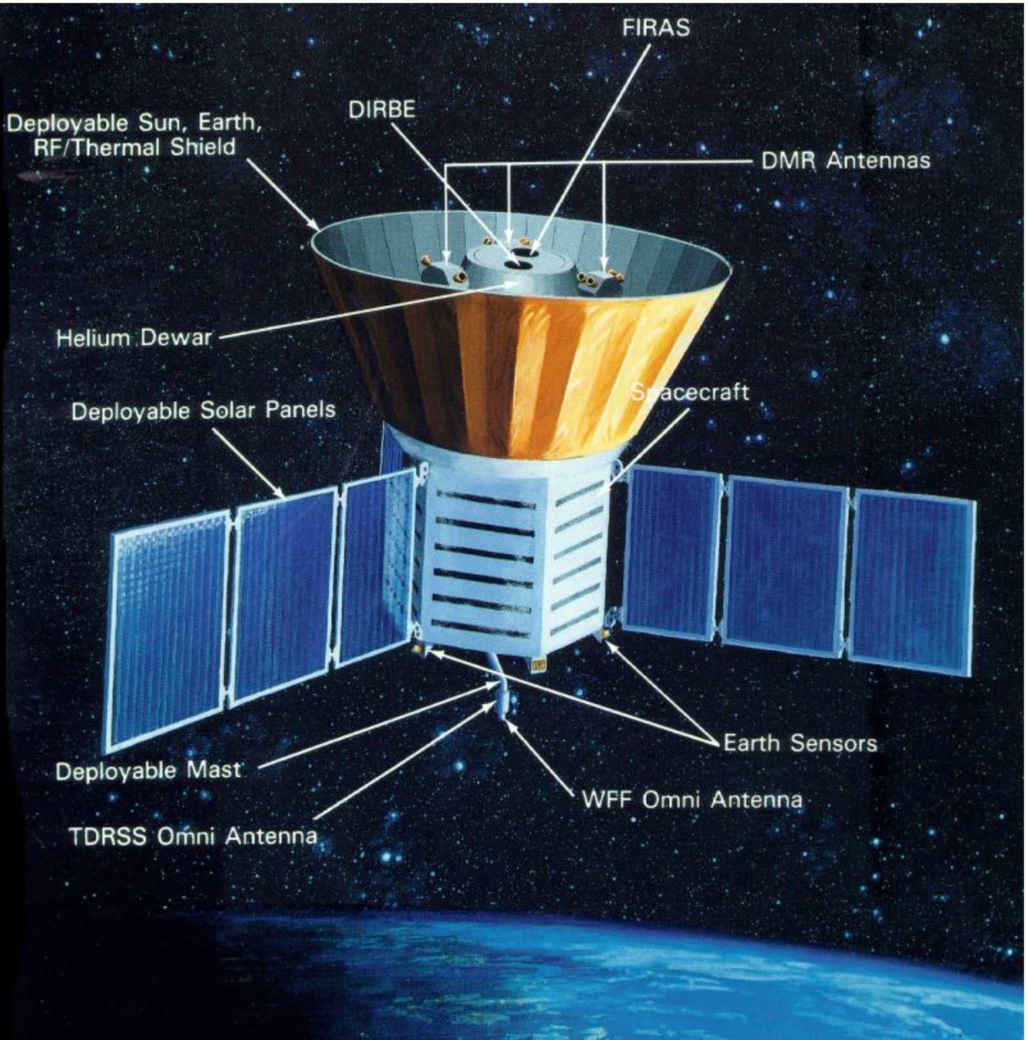
$I/\nu^3 \sim n/\nu^2$ lorentz invariant

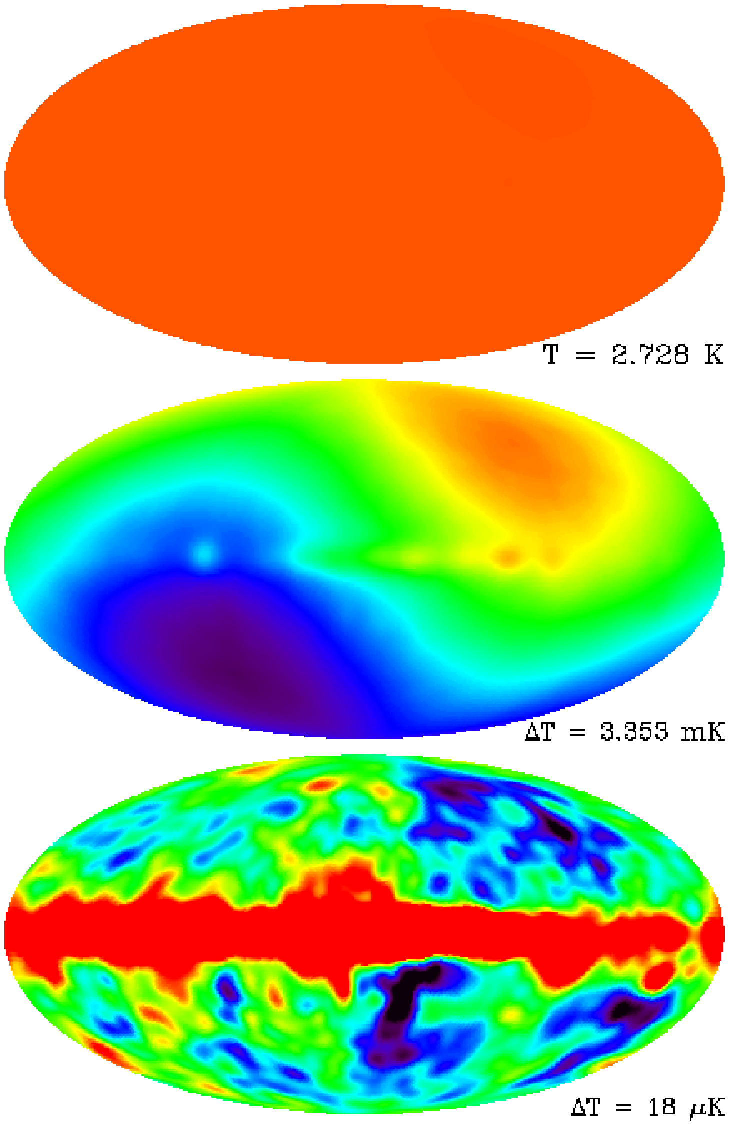
$$\frac{T'}{T} = \frac{\nu'}{\nu}$$



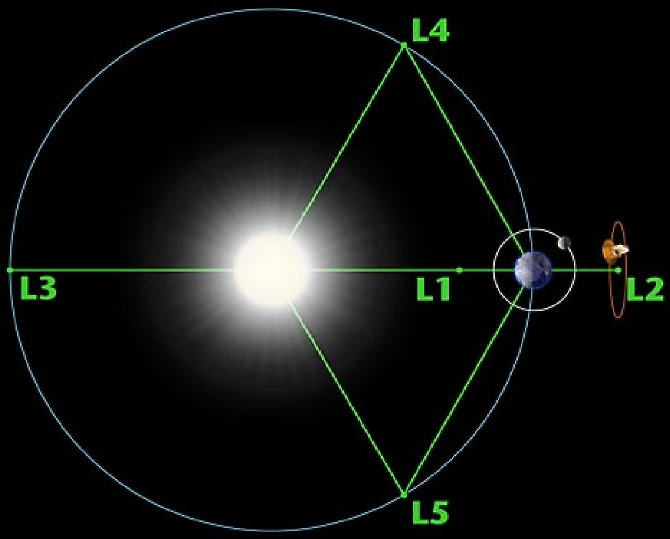
# COBE

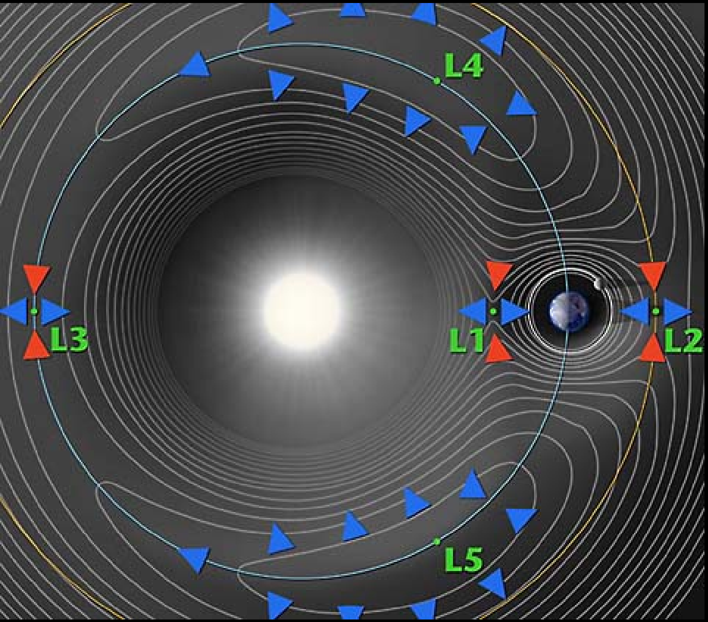
start 1989

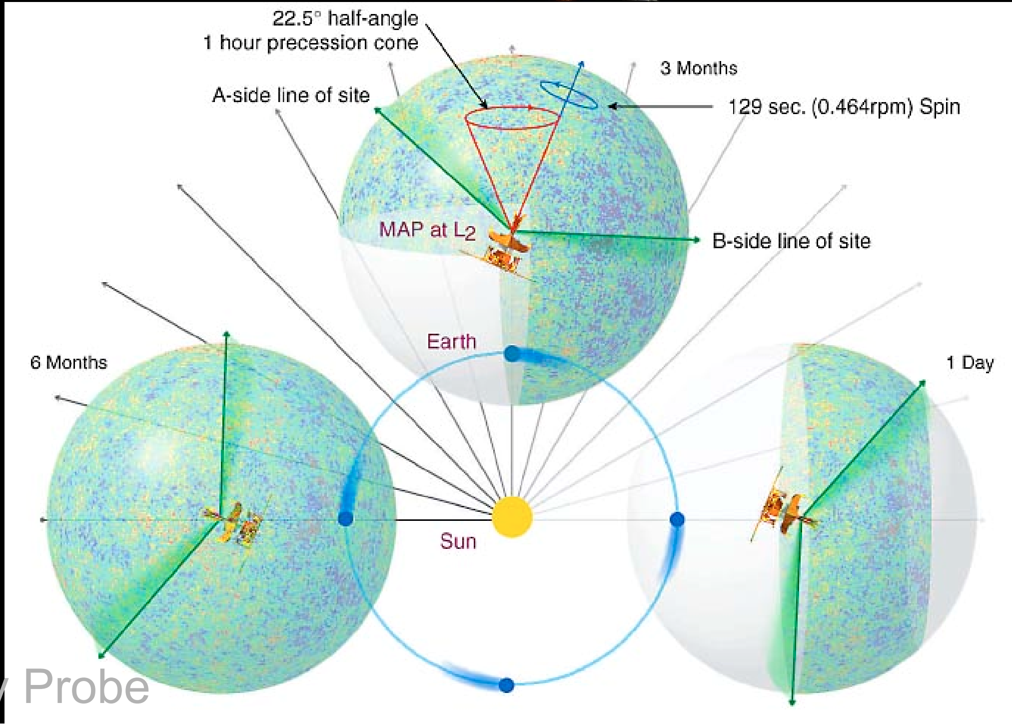


# WMAP*




* WMAP=Wilkinson Microwave Anisotropy Probe

# WMAP Data

a CMB* sky map $\Delta T(\theta,\phi)$
is a tabulated function
of the difference $\Delta T$ between
local CMB temperature at $(\theta,\phi)$ and mean CMB temp.

derived from pairs $T(\theta_i,\phi_i)-T(\theta_j,\phi_j)$

* CMB=Cosmic Microwave Background

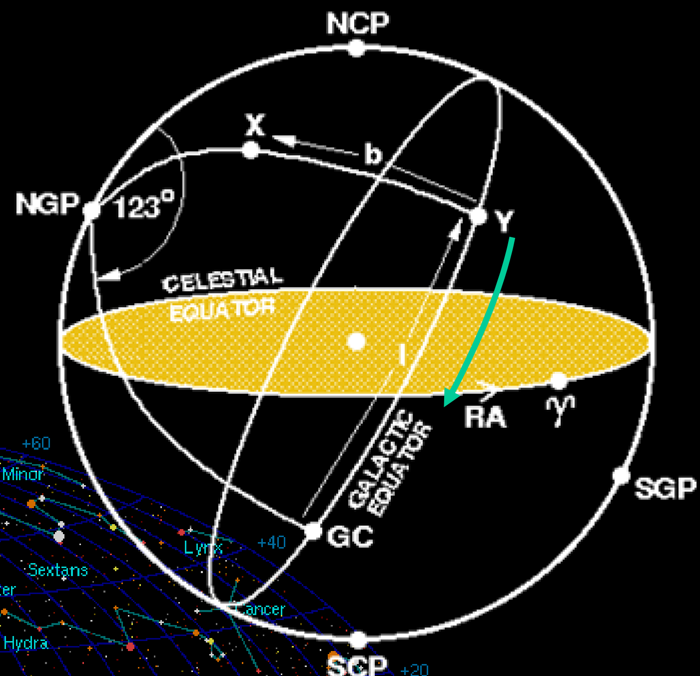
# Galactic Coordinate System

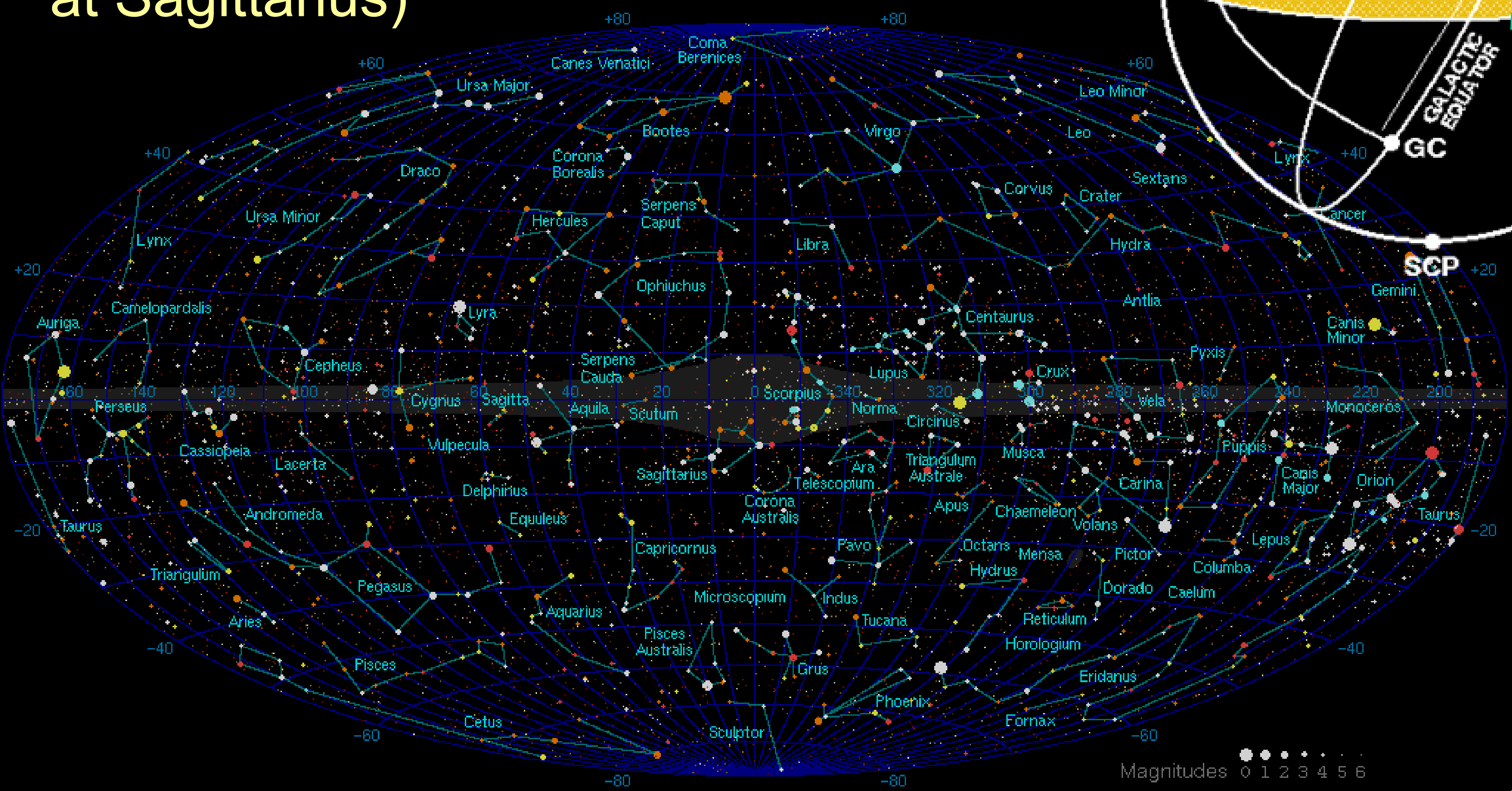
$90° - \theta = b$ = galactic latitude (Breite)

($\theta=0$: NGP, $\downarrow$antiaxis of galaxy rotation)

$\phi = l$ = galactic longitude (Länge)

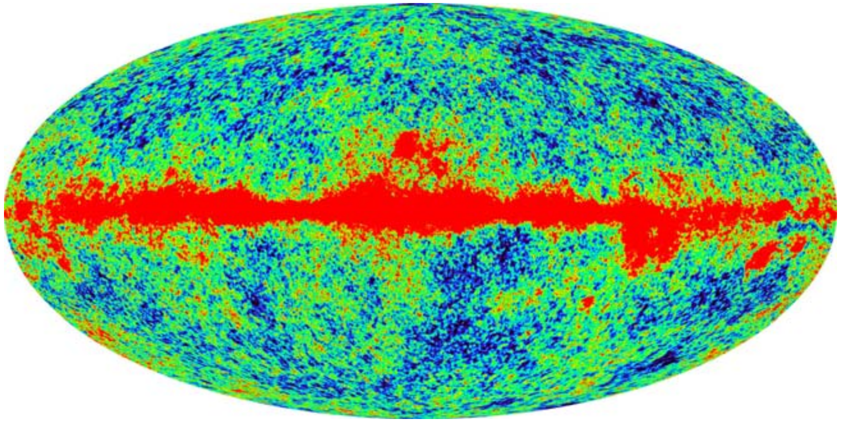
($\phi=0$: $\rightarrow$ GC = centre of galaxy, at Sagittarius)



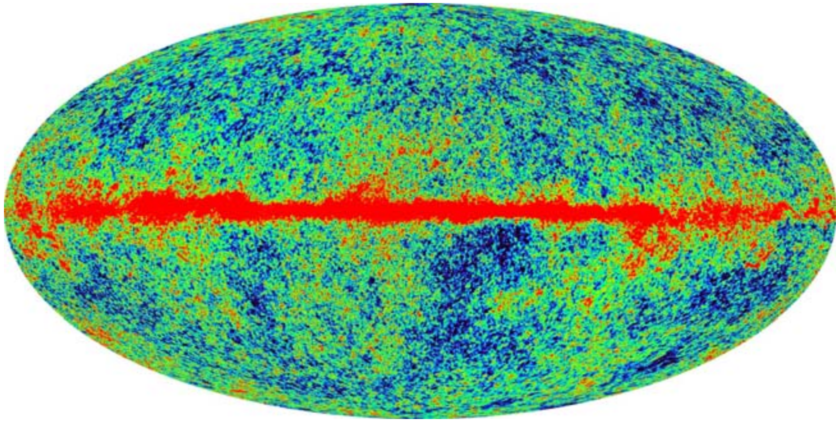



# WMAP Sky Maps (2001-03)

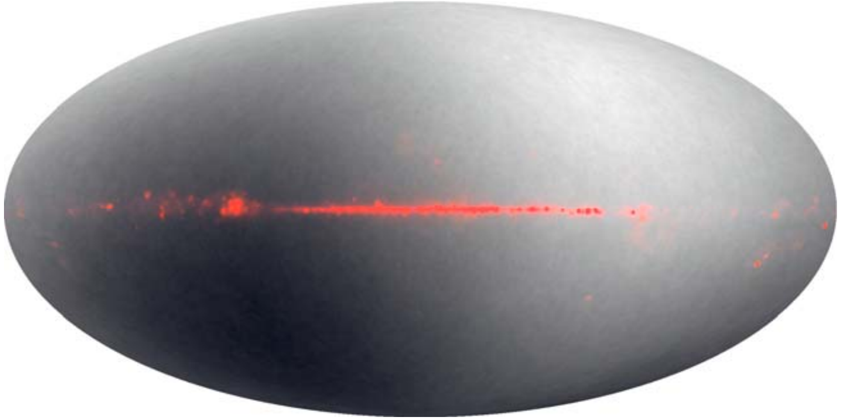
Q-Band (41GHz/7.3mm)



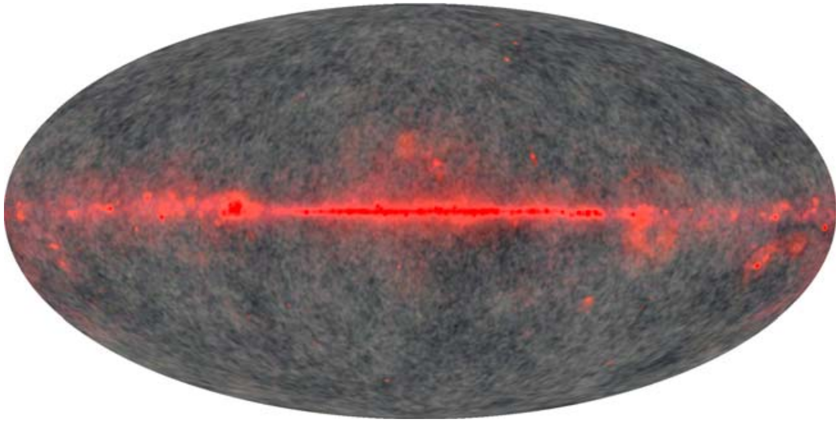
W-Band (94GHz/3.2mm)



QVW→RGB incl. dipole



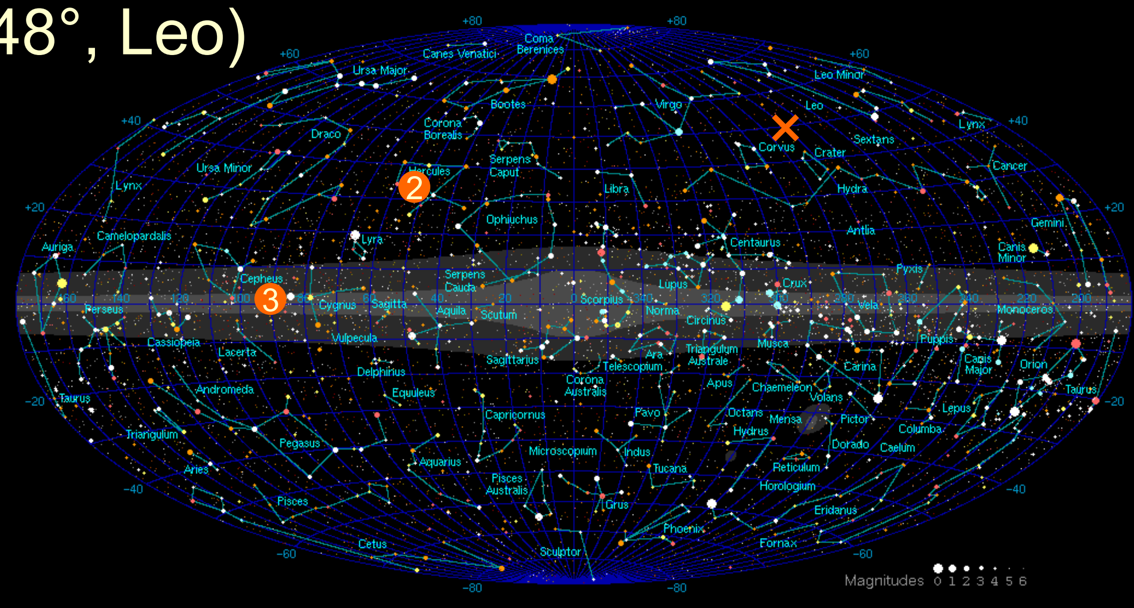
dipole subtracted



# Movement of Earth

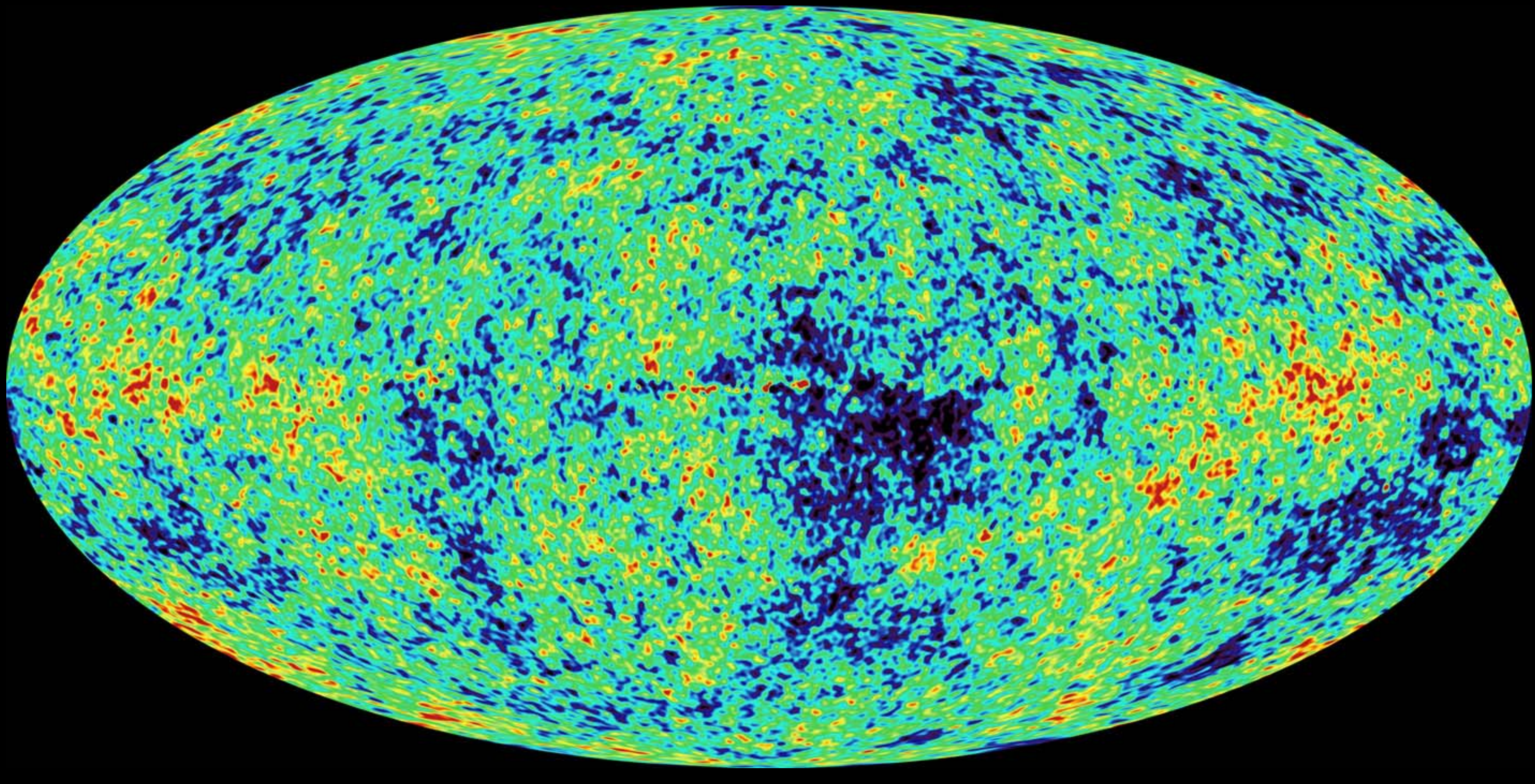
1. orbit: 30 km/s (direction in ecliptic)
2. sun rel. to local stars: 20 km/s (dir. Hercules)
3. milky way rotation: 220 km/s (dir. $\phi$=90°, b=0, Cygnus)
4. milky way rel. to centre of local cluster: 40 km/s
5. local cluster rel. to CMB: 600 km/s

result: × 370 km/s = CMB dipole
(dir. $\phi$=264°, b=+48°, Leo)



# WMAP Sky Maps: $\Delta T$

weighted linear combination of 5 Bands (23–94 GHz)
to eliminate galactic foreground range ±200μK



# The Early Universe is not Transparent

Plasma
(nuclei + electrons
 + photons)

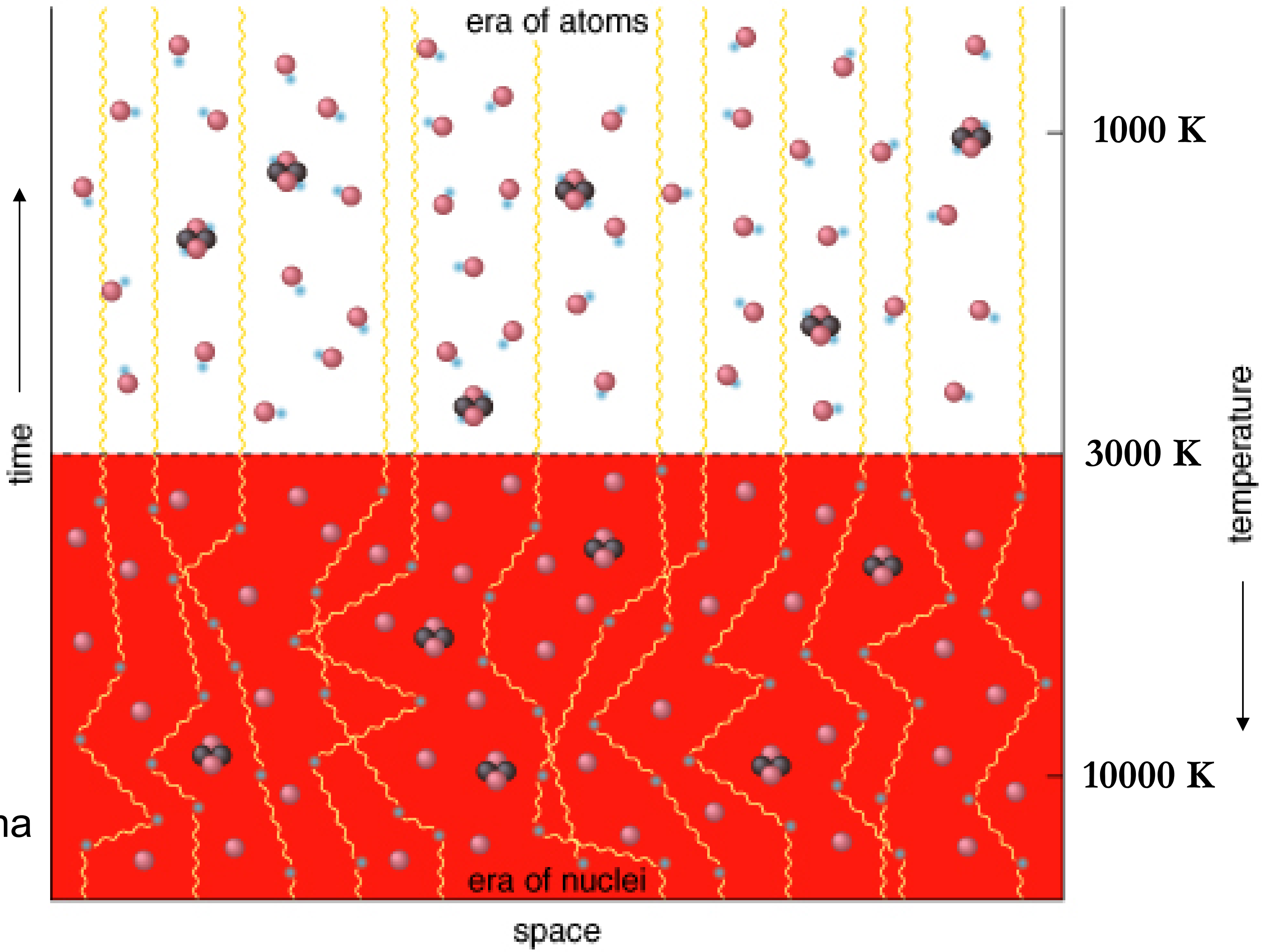

at age = 300 000 years
neutral atoms (gas)

# Photon/Matter-Decoupling

Age
ca.
300 000 a
$10^{13}$ s

1+z=1090

plasma



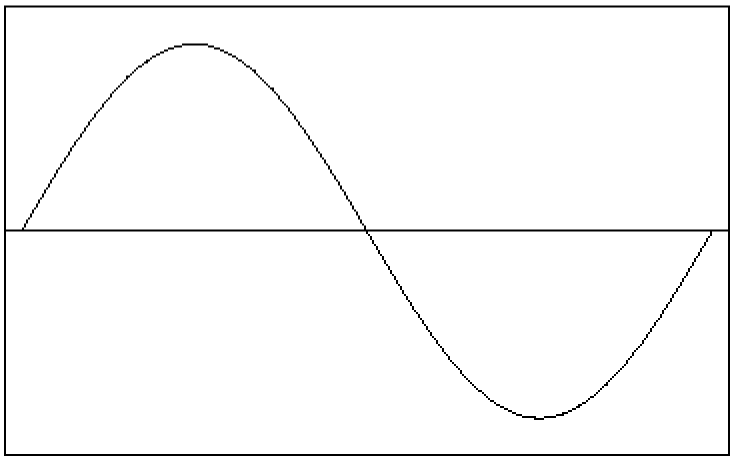
# Structure in Background Radiation…

…shows random acoustic waves in the fluid

of matter (baryons) and radiation (photons)

that filled the universe at recombination time

# Harmonic Analysis: Fourier




$$f(\phi) = \sum_{l=1}^{19} a_l \sin l\phi$$

function on a circle,
$\phi = 0 \ldots 2\pi$

# Data Analysis

spherical Fourier analysis (function on a sphere)
= expansion in spherical harmonics = multipoles

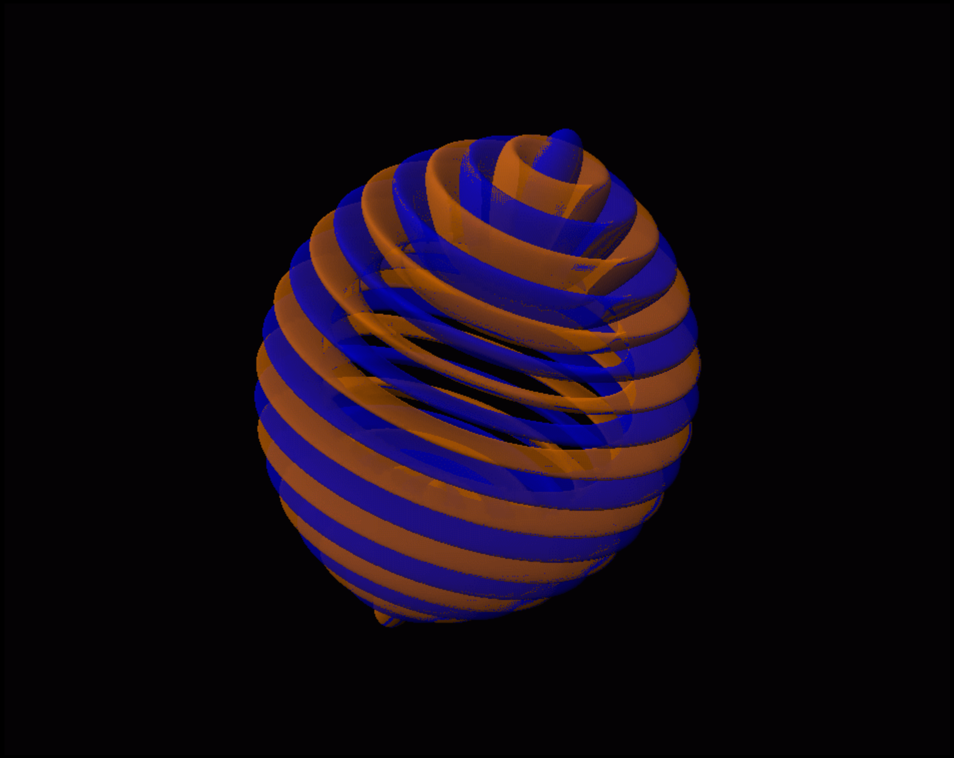
$$\Delta T(\theta,\phi) = \sum_{l=2}^{800} \sum_{m=-l}^{l} a_{lm} Y_l^m(\theta,\phi)$$

complete set of orthonormal functions on the sphere
$l$ = 0: mean T, not measured (mean ΔT=0)
$l$ = 1: dipole moment = Doppler shift (removed from data)

$$a_{lm} = \int_{\text{sphere}} \Delta T(\theta,\phi) \cdot Y_l^{m*}(\theta,\phi)\ \mathrm{d}\Omega$$

# Example: $l$ = 19

$m = -19 \ldots +19$
$l$ = measure of "granularity"

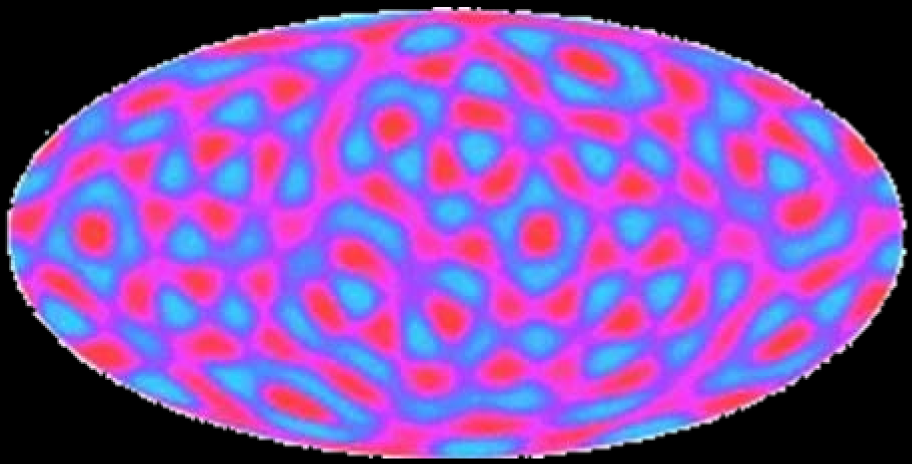


$\dfrac{\pi}{l} \approx \Delta\alpha$ = angular size of cells of temperature fluctuations

# Example: $l$ = 16

random mixture of *m*

$l$ = measure of "granularity" $\quad \Delta\alpha \approx 10°$



$$\frac{\pi}{l} \approx \Delta\alpha$$ = angular size of cells of temperature fluctuations

# Rotational (In)variance

if we rotate the coordinate system $\theta,\phi \to \theta',\phi'$,
the $a_{lm}$ are transformed,
but the norm for any $l$ is invariant:

$$\Delta T(\theta,\phi) = \sum_{l=2}^{800} \sum_{m=-l}^{l} a_{lm} Y_l^m(\theta,\phi)$$

$$\Delta T(\theta',\phi') = \sum_{l=2}^{800} \sum_{m=-l}^{l} a'_{lm} Y_l^m(\theta',\phi')$$

$$c_l := \sum_{m=-l}^{l} | a'_{lm} |^2 = \sum_{m=-l}^{l} | a_{lm} |^2$$

(cf. angular momentum physics, multipole radiation)

# Stochastic Universe

CMB temp. fluctuations are Gaussian

$\Rightarrow \langle a_{lm} \rangle = 0$

$\Rightarrow \sigma^2(a_{lm}) = \langle |a_{lm}|^2 \rangle = C_l \neq 0$ depend on cosmological model

$\Rightarrow \mathrm{cov}(a_{lm}, a_{l'm'}) = \langle a_{lm} \cdot a^*_{l'm'} \rangle = 0$

then the only relevant numbers are

$$C_l = \frac{1}{2l+1} \sum_{m=-l}^{l} | a_{lm} |^2$$

set of $C_l$ = power spectrum

# Results

power spectrum
per log $l$
T T = temperature

T E = polarisation