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On the second part of Hilbert's 16th problem

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Abstract

Let k be an integer such that k is larger than or equal to zero, and let H be the Hilbert number. In this paper, we use the method of describing functions to prove that in the Liénard equation, the upper bound for $H(2k+1)$ is k. By applying this method to any planar polynomial vector field, it is possible to completely solve the second part of Hilbert's 16th problem.

Keywords: Second part of Hilbert's 16th problem; Hilbert number; Liénard equation; Describing function; Limit cycle; Polynomial vector field

1. Introduction

In 1900, Hilbert presented a list consisting of 23 mathematical problems (see [1]). The second part of the 16th problem appears to be one of the most persistent in that list, second only to the 8th problem, the Riemann conjecture. The second part of the 16th problem is traditionally split into three parts (see [2]).

Problem 1. *A limit cycle is an isolated closed orbit. Is it true that a planar polynomial vector field has but a finite number of limit cycles*?

Problem 2. *Is it true that the number of limit cycles of a planar polynomial vector field is bounded by a constant depending on the degree of the polynomials only*?

Denote the degree of the planar polynomial vector field by n. The bound on the number of limit cycles in Problem 2 is denoted by $H(n)$, and is known as the Hilbert number. Linear vector fields have no limit cycles, hence $H(1)=0$.

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Problem 3. *Give an upper bound for* $H(n)$.

Let k be an integer such that $k \geqslant 0$. In 1977, Lins et al. [3] found examples with k different limit cycles in the Liénard equation

$$\dot{x} = y - F(x),$$
$$\dot{y} = -x, \tag{1}$$

where

$$F(x) = q_{2k+1}x^{2k+1} + q_{2k}x^{2k} + \cdots + q_2x^2 + q_1x.$$

The degree of this polynomial vector field is denoted by $2k+1$. The coefficients q_i (for integers i such that $1 \leqslant i \leqslant 2k+1$) are real constants. Lins et al. [3] conjectured the number k as the upper bound for the number of limit cycles of the Liénard equation (1). Their conjecture thus states that in the Liénard equation (1), the upper bound for $H(2k+1)$ is k.

In his list of mathematical problems for the next century, published in 1998, Smale [4] mentioned the Liénard equation (1) as a simplified version of the second part of Hilbert's 16th problem (see [4]).

In the present paper, we will prove the conjecture stated by Lins et al. [3] in 1977, thereby solving the simplified version of the second part of Hilbert's 16th problem stated by Smale [4] in 1998.

2. Preliminaries

In this section, we will introduce the method of describing functions, which may be used to calculate limit cycles in nonlinear dynamic systems (see [4]).

Consider a dynamic system

$$\dot{x} = \boldsymbol{Mx} + \boldsymbol{h}(\boldsymbol{x}),$$

where $\boldsymbol{x}$ is the m-dimensional vector of state variables, $\boldsymbol{M}$ is an $m \times m$ constant matrix and $\boldsymbol{h}(\boldsymbol{x})$ is an m-dimensional vector of nonlinear functions.

Assume that the state variables are dominated by a harmonic term of a specific order

$$\boldsymbol{x} \cong \boldsymbol{a}_0 + \boldsymbol{a}_1 \sin(\omega t),$$

where $\boldsymbol{a}_0$ is the m-dimensional vector of center values, $\boldsymbol{a}_1$ is the m-dimensional vector of amplitudes and ω is the frequency. $\boldsymbol{a}_0$, $\boldsymbol{a}_1$ and ω are assumed to be real. $\boldsymbol{a}_1$ and ω are nonzero.

Then, approximate the vector of nonlinear functions by discarding higher harmonic terms (terms of the form $\cos(r\omega t)$ and $\sin(r\omega t)$ for integers r such that $r \geqslant 2$)

$$\boldsymbol{h}(\boldsymbol{x}) \cong \boldsymbol{\Phi} + \boldsymbol{Na}_1 \sin(\omega t),$$

where $\boldsymbol{\Phi}$ is an m-dimensional constant vector and $\boldsymbol{N}$ is an $m \times m$ constant matrix. The components of $\boldsymbol{N}$ are called describing functions.

The system becomes

$$\dot{x} = \boldsymbol{M}\boldsymbol{a}_0 + \Phi + (\boldsymbol{M} + \boldsymbol{N})\boldsymbol{a}_1 \sin(\omega t)$$

and solutions for $\boldsymbol{a}_0$, $\boldsymbol{a}_1$ and ω satisfy

$$\boldsymbol{M}\boldsymbol{a}_0 + \Phi = 0, \tag{2}$$

$$\det(j\omega\boldsymbol{I} - \boldsymbol{M} - \boldsymbol{N}) = 0, \tag{3}$$

where $\boldsymbol{I}$ is the $m \times m$ identity matrix and j satisfies the equation $j^2 = -1$.

3. Result

In this section, we will prove the conjecture stated by Lins et al. [3] in 1977, by applying the method of describing functions to the Liénard equation (1).

Theorem. *Let k be an integer such that $k \geqslant 0$, and let H be the Hilbert number. For the Liénard equation* (1), *we have that the upper bound for $H(2k+1)$ is k.*

Proof. Noticing that the state variable x of the Liénard equation (1) behaves approximately like a sine function in simulations (see Fig. 1), we assume—in order to make a good approximation of x—that both state variables are dominated by a harmonic term of a specific order

$$\begin{bmatrix} x \\ y \end{bmatrix} \cong \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} + \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \sin(\omega t) \tag{4}$$

which gives that the Liénard equation (1) becomes

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix} = \begin{bmatrix} -q_1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} + \begin{bmatrix} -q_1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \sin(\omega t)$$

$$+ \begin{bmatrix} -q_{2k+1}[a_0 + a_1 \sin(\omega t)]^{2k+1} - q_{2k}[a_0 + a_1 \sin(\omega t)]^{2k} - \cdots - q_2[a_0 + a_1 \sin(\omega t)]^2 \\ 0 \end{bmatrix}.$$

Since the nonlinear function only affects $f(t)$, Eq. (2) gives that the constant part of $g(t)$ may be set equal to zero at this stage, so that $a_0 = 0$.

The nonlinear function in $f(t)$ becomes

$$\begin{aligned} &-q_{2k+1}x^{2k+1} - q_{2k}x^{2k} - \cdots - q_3x^3 - q_2x^2 \\ &\quad = -q_{2k+1}a_1^{2k+1}\sin^{2k+1}(\omega t) - q_{2k}a_1^{2k}\sin^{2k}(\omega t) - \cdots \\ &\qquad - q_3a_1^3\sin^3(\omega t) - q_2a_1^2\sin^2(\omega t) \end{aligned}$$

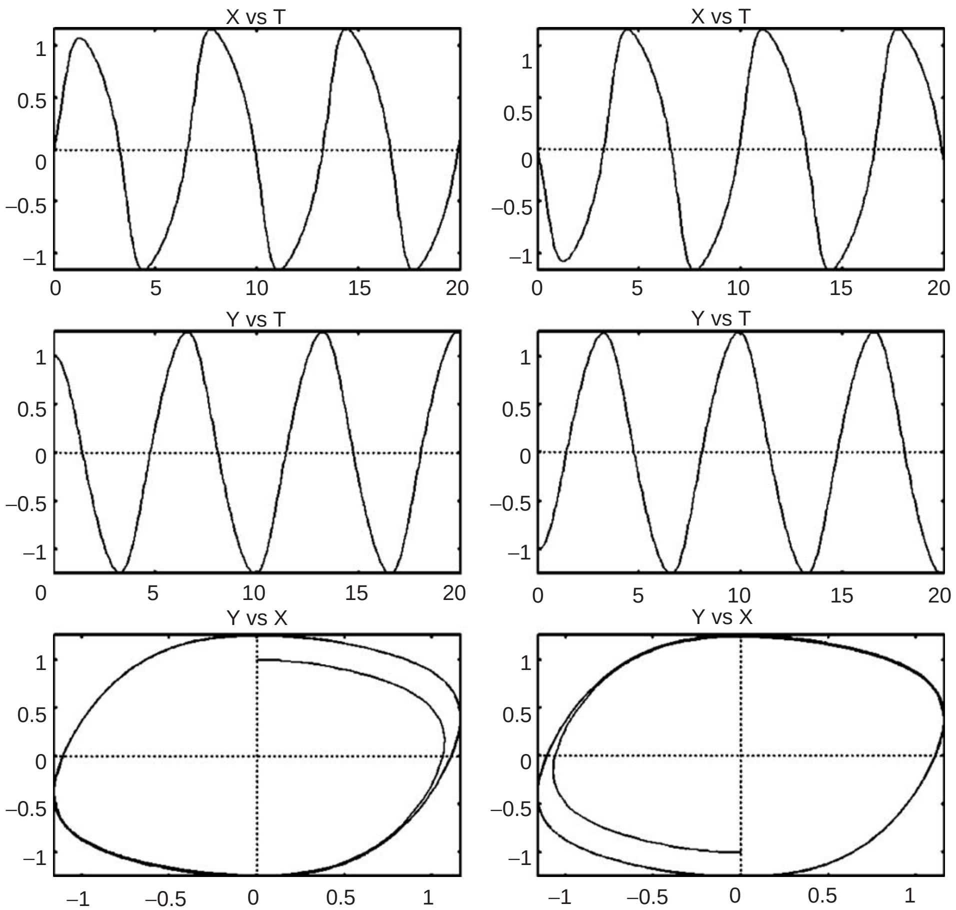

Fig. 1. Numerical integration of the Liénard equation (1) when $F(x) = x^3 - x$. In the left figures, the initial conditions are $x(0) = 0$ and $y(0) = 1$. In the right figures, the initial conditions are $x(0) = 0$ and $y(0) = -1$.

$$
\begin{aligned}
= &-\frac{q_{2k}a_1^{2k}[1+\cos(2\omega t)]^k}{2^k} - \frac{q_{2k-2}a_1^{2k-2}[1+\cos(2\omega t)]^{k-1}}{2^{k-1}} - \cdots \\
&-\frac{q_2 a_1^2[1+\cos(2\omega t)]}{2} \\
&+\left[-\frac{q_{2k+1}a_1^{2k+1}[1+\cos(2\omega t)]^k}{2^k} - \frac{q_{2k-1}a_1^{2k-1}[1+\cos(2\omega t)]^{k-1}}{2^{k-1}} - \cdots \right. \\
&\left. -\frac{q_3 a_1^3[1+\cos(2\omega t)]}{2}\right] \sin(\omega t)
\end{aligned}
$$

$$\cong -q_{2k}\alpha_k a_1^{2k} - q_{2k-2}\alpha_{k-1}a_1^{2k-2} - \cdots - q_2\alpha_1 a_1^2$$
$$+[-q_{2k+1}\alpha_k a_1^{2k+1} - q_{2k-1}\alpha_{k-1}a_1^{2k-1} - \cdots - q_3\alpha_1 a_1^3]\sin(\omega t)$$

such that each α_l always is larger than or equal to $1/2^l$, for integers l such that $1 \leqslant l \leqslant k$.

The Liénard equation (1) becomes

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -q_1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} + \begin{bmatrix} -q_{2k}\alpha_k a_1^{2k} - q_{2k-2}\alpha_{k-1}a_1^{2k-2} - \cdots - q_2\alpha_1 a_1^2 \\ 0 \end{bmatrix}$$
$$+ \left(\begin{bmatrix} -q_1 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} -q_{2k+1}\alpha_k a_1^{2k} - \cdots - q_3\alpha_1 a_1^2 & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \sin(\omega t).$$

Letting the constant part of $f(t)$ equal to zero according to Eq. (2) gives that

$$b_0 = q_{2k}\alpha_k a_1^{2k} + q_{2k-2}\alpha_{k-1}a_1^{2k-2} + \cdots + q_2\alpha_1 a_1^2 \tag{5}$$

so that the coefficients in front of the even powers of the polynomial $F(x)$ only have an impact on the center value b_0 (see also Example 1).

Eq. (3) gives that the solutions for a_1 and ω satisfy

$$\omega = \pm 1 \tag{6}$$

so that the dominant harmonic order is the first one, and

$$q_{2k+1}\alpha_k a_1^{2k} + \cdots + q_3\alpha_1 a_1^2 + q_1 = 0 \tag{7}$$

which has at most k distinct zeros in terms of a_1^2. For each such zero a_1^2, we have that $\pm a_1$ are solutions.

The approximation of the state variables as in Eq. (4) fits the solution in terms of x. To solve for y, we notice that $\dot{y} = -x$ from the Liénard equation (1), which gives that

$$y = b_0 + \frac{a_1 \cos(\omega t)}{\omega} \tag{8}$$

since x is as in Equation (4) and ω is nonzero. Notice that b_0 still satisfies Eq. (5), since we would get that same equation if we assumed that the state variables were cosine functions instead of sine functions in Eq. (4).

Thus, the following are the possible cases for the state variables.

A. If a_1 and ω are of the same sign,

$$x \cong a_1 \sin t,$$

$$y \cong b_0 + a_1 \cos t.$$

B. If a_1 and ω are of different signs

$$x \cong -a_1 \sin t,$$

$$y \cong b_0 - a_1 \cos t.$$

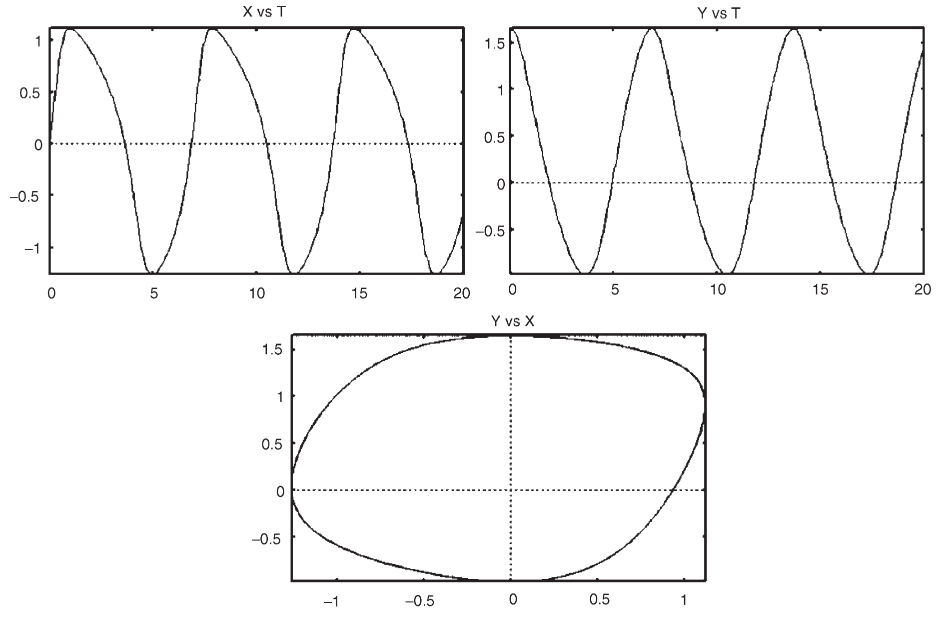

Fig. 2. Numerical integration of the Liénard equation (1) when $F(x)=x^3+0.5x^2-x$ and the initial condition $y(0)$ is positive.

For each distinct zero a_1^2 of Eq. (7), these two cases correspond to the same limit cycle. Simulations show that the sign of the initial condition $y(0)$ determines whether the trajectory follows the solution of case A or case B (see Fig. 1).

We thus have that for each unique x that may be approximated as in Eq. (4), there exists a unique y as in Eq. (8). Therefore, we have that there exist at most k distinct limit cycles in the Liénard equation (1).

In Eq. (4), we assumed that the state variables were dominated by a harmonic term of a specific order. If this assumption is not true, it is possible to increase the accuracy of the approximation of the state variables by adding higher harmonic terms to Eq. (4). By doing this, and by going through the calculations one more time, we would—if the state variables were not dominated by a harmonic term of a specific order—end up with other amplitudes than in the first calculation. It is of great importance for the result to understand that this change in amplitudes does not mean that there exist additional limit cycles in the Liénard equation (1)—it only means that the new approximation of the state variables has a higher accuracy (see also [5]).

Thereby, we have proved that the maximum number k of limit cycles that exist in the Liénard equation (1), depends on the degree $2k+1$ of the polynomial $F(x)$ only. Hence, we have proved that the upper bound for $H(2k+1)$ is k.

Note that the method of describing functions may be used in a similar manner as in the proof above, to find the upper bounds for the Hilbert number in any planar polynomial vector field. Thus, it is possible to completely solve the second part of Hilbert's 16th problem by using this approach.

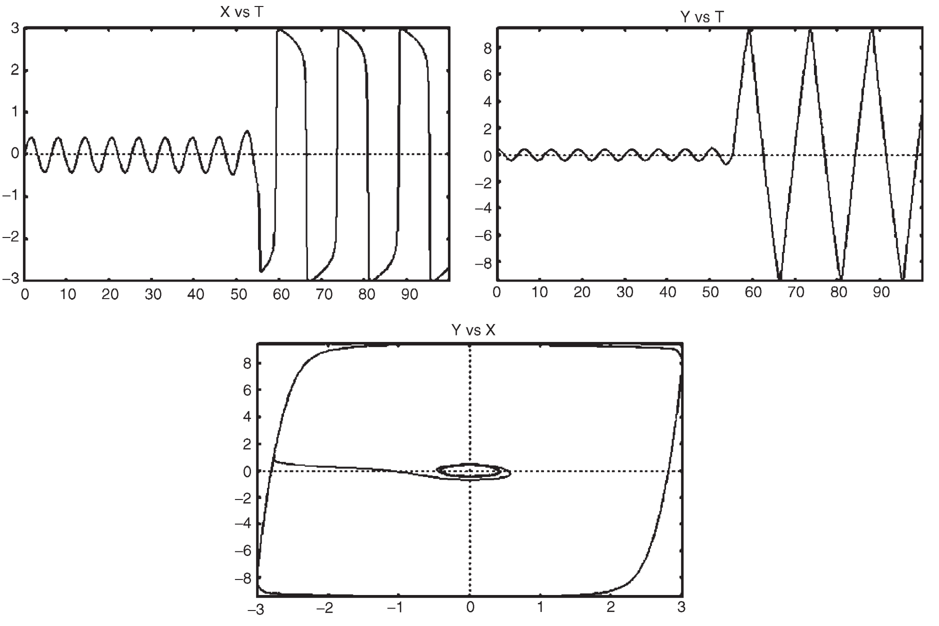

Fig. 3. Numerical integration of the Liénard equation (1) when $F(x) = 0.25x^5 - 2x^3 + 0.25x$ and the initial condition $y(0)$ is positive. The initial conditions lie just outside the small cycle, and the trajectory is attracted to it at first. Then, at $t = 55$, it becomes attracted to the large cycle.

Example 1. Let $F(x) = x^3 + 0.5x^2 - x$ (Fig. 2).
Eqs. (5)–(8) give that

$$x \cong \pm 1.41 \sin t, \qquad y \cong 0.5 \pm 1.41 \cos t.$$

Example 2. Let $F(x) = 0.25x^5 - 2x^3 + 0.25x$ (Fig. 3).
Eqs. (5)–(8) give that

$$x \cong \begin{cases} \pm 3.23 \sin t, \\ \pm 0.51 \sin t, \end{cases}$$

$$y \cong \begin{cases} \pm 3.23 \cos t, \\ \pm 0.51 \cos t. \end{cases}$$

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